# UNIVERSITY OF OSLO

## Faculty of Mathematics and Natural Sciences

Examination in: STK2130 — Modelling by stochastic processes

Day of examination: Thursday June 6th 2019

Examination hours: 14.30 – 18.30

This problem set consists of 3 pages.

Appendices: None

Permitted aids: Approved calculator, "List of formulas for STK1100 and STK1110 ".

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All parts of the problems count equally to the final grade.

## Problem 1

Consider a discrete time Markov chain with state space $\{1, 2, 3, 4, 5\}$ and matrix with one-step transition probabilities

$$\begin{bmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix}.$$

a) Explain what the classes are. Which are recurrent and which are transient?

b) Find the element $P^2_{4,5}$, i.e the probability that the chain is in state 5 after two steps when it starts in state 4.

c) Compute $P(X_3 = 2, X_k \notin \{1, 3, 5\}, k = 1, 2 | X_0 = 4)$.

d) Compute $P(X_3 = 1, X_k \notin \{1, 3, 5\}, k = 1, 2 | X_0 = 4)$.

e) Let the random variable T be $T = \min\{n \geq 0 | X_n \in \{1, 3, 5\}\}$ and define the conditional expectations $\mu_i = E[T | X_0 = i], i = 1, \ldots, 5$.

   Find $\mu_i, i = 1, \ldots, 5$.

f) Explain why $\pi_5 = \lim_{n\to\infty} P(X_n = 5 | X_0 = 1)$ exists. What is $\pi_5$?

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# Problem 2

a) Explain how a birth and death process with parameters $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ is defined.

Consider a service station where customers arrive in accordance with a Poisson process having rate $\lambda$. There are $s$ servers and an entering customer goes to a free server if one is available. If no one is free, the customer leaves the service station. The successive service times for each server are distributed as independent exponential random variables with mean/expectation $1/\mu$.

The technical description of such stations is a M/M/s/s queue, where s denotes that there are s servers and it is only possible with s customers. In other words there is no waiting room.

b) Explain why this service station can be described as a birth and death process and what the instantaneous transition rates are.

Recall that the Kolmogorov's backward equations have the form $P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$.

c) What are the Kolmogorov's backward equations that the transition probabilities $P_{ij}(t)$ satisfy in this system?

d) Explain what the balance equations to determine the limiting probabilities are in this case.

e) Find the limiting probabilities for this Markov chain.

# Problem 3

Consider a simple random walk, i.e. a Markov chain whose state space consists of the integers $0, \pm 1, \pm 2, \ldots$ and has transition probabilities given by

$$\begin{aligned} P_{i,i+1} &= p \\ P_{i,i-1} &= q = 1 - p \end{aligned}$$

where $0 < p < 1$. The other transition probabilities are 0.

a) Explain why the n-step probabilities are

$$\begin{aligned} P_{00}^{2n} &= \binom{2n}{n} p^n (1-p)^n \\ P_{00}^{2n-1} &= 0 \end{aligned}$$

for $n = 1, 2, \ldots$

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b) Show that $P_{00}^{2n}$ is approximately $\frac{[4p(1-p)]^n}{\sqrt{\pi n}}$.

Explain why the chain is recurrent when $p = 1/2$ and transient when $p \neq 1/2$.

Recall that Stirlings formula is $n! = (\frac{n}{e})^n \sqrt{2\pi n}$

In the rest of this problem we assume that $p < q$.
Define the random variables $T_i = \inf\{n \geq 1 : X_n = i\}$.

c) Let $f = P(T_1 < \infty | X_0 = 0)$. Why does $f$ satisfy the quadratic equation $f = p + qf^2$? The quadratic equation has two solutions 1 and $p/q$. Why is $p/q$ the reasonable value for f?

d) Using the result from part c) find $P(T_0 < \infty | X_0 = 0)$.

END