UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK2130 — Modelling by stochastic processes

Day of examination: Wednesday May 27th 2020.

Examination hours: May 27th, 14.30 – June 4th, 14.30

This problem set consists of 5 pages.

Appendices: None.

Permitted aids: All available notes and books.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Consider a discrete-time Markov chain $\{X_n : n \geq 0\}$ with state space $\mathcal{X} = \{1, 2, 3, 4\}$, and transition probability matrix:

$$\boldsymbol{P} = \begin{bmatrix} p & 0 & 0 & q \\ q & p & 0 & 0 \\ 0 & 0 & p & q \\ 0 & 0 & q & p \end{bmatrix}$$

where $0 < p < 1$, $0 < q < 1$ and $p + q = 1$.

a) Describe the Markov chain by a diagram.

b) The chain has three classes, $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{2\}$ and $\mathcal{C}_3 = \{3, 4\}$. For each of these classes discuss whether the class is *transient* or *recurrent*.

c) Show that the two-step transition probability matrix is given by:

$$\boldsymbol{P}^{(2)} = \begin{bmatrix} p^2 & 0 & q^2 & 2pq \\ 2pq & p^2 & 0 & q^2 \\ 0 & 0 & p^2 + q^2 & 2pq \\ 0 & 0 & 2pq & p^2 + q^2 \end{bmatrix}$$

d) Conditioned upon the chain has entered one of the states 3 or 4 find the stationary distribution over these two states.

e) We assume that $X_0 = 1$, and let M be given by:

$$M = \min\{m > 0 : X_m = 4\}$$

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Thus, M is the number of steps until the Markov chain enters state 4 for the first time given that the chain starts out in state 1. Show that the probability distribution of M is given by:

$$P(M = m) = p^{m-1}q, \quad m = 1, 2, \ldots$$

f) Find $E[M]$.

g) In the remaining part of this problem we assume that $X_0 = 2$, and let N be given by:

$$N = \min\{n > 0 : X_n = 3\}$$

Thus, N is the number of steps until the Markov chain enters state 3 for the first time given that the chain starts out in state 2. Find $E[N]$.

h) Find the probability distribution of N.

Problem 2

An urn always contains 2 balls. The balls are colored either *red* or *blue.* At each stage a ball is randomly chosen and then replaced by a new ball, which with probability 0.75 is the *same color*, and with probability 0.25 is the *opposite color*, as the ball it replaces. This is modelled by a Markov chain $\{X_n : n \geq 0\}$ where:

$X_n =$ The number of *red* balls after the nth selection. $\quad n = 0, 1, 2, \ldots$

Thus, the state space of the Markov chain is $\mathcal{X} = \{0, 1, 2\}$.

a) Explain why the transition probability matrix of this Markov chain is:

$$\boldsymbol{P} = \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.125 & 0.75 & 0.125 \\ 0 & 0.25 & 0.75 \end{bmatrix}$$

b) It can be calculated that:

$$\boldsymbol{P}^{(4)} \approx \begin{bmatrix} 0.4238 & 0.4688 & 0.1074 \\ 0.2344 & 0.5313 & 0.2344 \\ 0.1074 & 0.4688 & 0.4238 \end{bmatrix}$$

You do not need to calculate this.

Find the probability that the fifth ball selected is *red* given that $X_0 = 2$.

c) Find the stationary distribution for the Markov chain $\{X_n : n \geq 0\}$.

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d) Let ρ_n denote the probability that the nth ball selected is *red* given that $X_0 = 2$. Find:

$$\lim_{n\to\infty} \rho_n$$

e) We now introduce:

$$N_j = \min\{n > 0 : X_n = j\}, \quad j \in \mathcal{X}.$$

Thus, N_j is the number of steps until the Markov chain makes a transition into state j. We then let:

$$m_j = E[N_j | X_0 = j], \quad j \in \mathcal{X}.$$

That is, m_j is the expected number of steps until the Markov chain returns to state j given that it starts out in state j.

Find m_j for all $j \in \mathcal{X}$.

Problem 3

In this problem we consider a population consisting of individuals able to produce offspring of the same kind. We assume that each individual will, by the end of its lifetime, have produced r new offspring with probability p_r, $r = 0, 1, 2, \ldots$, independently of the numbers produced by other individuals. We assume that $p_0 > 0$, and that $p_r < 1$ for $r = 0, 1, 2, \ldots$.

The number of individuals initially present, denoted by X_0, is called the size of the 0-th generation. Moreover, we let:

$$X_n = \text{The population size in the } n\text{th generation}, \quad n = 0, 1, 2, \ldots$$

a) Explain why $\{X_n : n \geq 0\}$ is a *Markov chain*.

b) Explain why state 0 is a *recurrent state*, and why any state $j > 0$ is *transient*.

c) In the rest of this problem we assume that $X_0 = 1$, and $E[X_1] = \mu$. Show that $E[X_n] = \mu^n$.

d) We then consider the probability that the population eventually dies out:

$$\pi_0 = \lim_{n\to\infty} P(X_n = 0 \mid X_0 = 1)$$

Show that π_0 satisfies the following equation:

$$\pi_0 = \sum_{r=0}^{\infty} \pi_0^r \, p_r \qquad (1)$$

e) In the following you may use without proof that π_0 is the *smallest positive number* that satisfies (1).

Assume that $p_0 = \frac{1}{5}$, $p_1 = \frac{1}{5}$, $p_2 = \frac{3}{5}$, and that $p_r = 0$, for $r > 2$.

Calculate μ and π_0 in this case.

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Problem 4

A system can be in three possible states denoted respectively 0, 1 and 2. If the system is in state 0, it is considered to be *failed*, while if the system is in state 2, it is considered to be *functioning perfectly*. The state 1 represents an intermediate case where the system is functioning, but at a lower performance level than when it is in state 2.

We model this as a continuous-time Markov chain $\{X(t) : t \geq 0\}$ with state space $\mathcal{X} = \{0, 1, 2\}$. The system can transit from state i to state $i+1$ with rate μ, $i = 0, 1$. Such a transition is called a *repair*. Moreover, the system can transit from state i to state $i-1$ with rate λ, $i = 1, 2$. Such a transition is called a *failure*. Thus, a single repair can only increase the state by 1. Similarly, a single failure can only reduce the state by 1. It is *not possible* to transit directly from state 0 to state 2 or directly from state 2 to state 0. Finally, we assume that $\mu > 0$ and $\lambda > 0$.

We also introduce:

$$P_{ij}(t) = P(X(t) = j | X(0) = i), \quad \text{for all } i, j \in \mathcal{X}.$$

Moreover, for all $i, j \in \mathcal{X}$ we let:

$$q_{ij} = \text{The transition rate from state } i \text{ to state } j \text{ if } i \neq j.$$

$$v_i = \sum_{j \in \mathcal{X} \setminus i} q_{ij}.$$

Finally, we let the matrix $\boldsymbol{R}$ be given by:

$$\boldsymbol{R} = \begin{bmatrix} -v_0 & q_{0,1} & q_{0,2} \\ q_{1,0} & -v_1 & q_{1,2} \\ q_{2,0} & q_{2,1} & -v_2 \end{bmatrix}$$

a) Determine the matrix $\boldsymbol{R}$ expressed in terms of μ and λ.

b) Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2)$, where:

$$\pi_j = \lim_{t \to \infty} P_{ij}(t), \quad \text{for all } j \in \mathcal{X}.$$

Formulate a set of equations which can be used to determine $\boldsymbol{\pi}$, and solve these equations.

c) Assume that $X(0) = 2$, and let:

$$T = \min\{t > 0 : X(t) \neq 2\}$$

Explain briefly why we have:

$$P(T > t) = e^{-\lambda t}, \quad \text{for all } t > 0.$$

d) We still assume that $X(0) = 2$. However, we now consider the case where $\mu = 0$, and let:

$$S = \min\{t > 0 : X(t) = 0\}$$

What is the probability distribution of S? Explain your answer.

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Problem 5

Let $\{X(t) : t \geq 0\}$ be a standard Brownian motion process, and let $0 < t_1 < t_2$.

a) Find the joint density of $X_1 = X(t_1)$ and $X_2 = X(t_2)$.

b) Show that $(X_2|X_1 = x_1) \sim N(x_1, t_2 - t_1)$.

c) Show that $(X_1|X_2 = x_2) \sim N(\frac{t_1}{t_2}x_2, \frac{t_1}{t_2}(t_2 - t_1))$.

d) Find $P(\max_{0 \leq s \leq 4} X(s) \geq 2)$.

END