UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK2130 — Modelling by stochastic processes

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This problem set consists of 16 pages.

Appendices: None.

Permitted aids: All available notes and books.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

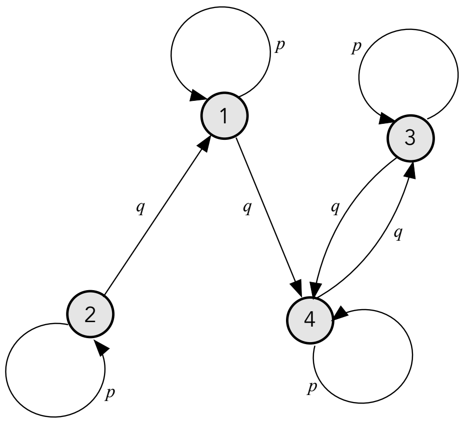

Figure 1: Diagram representing the Markov chain in Problem 1a

Consider a discrete-time Markov chain $\{X_n : n \geq 0\}$ with state space $\mathcal{X} = \{1, 2, 3, 4\}$, and transition probability matrix:

$$\boldsymbol{P} = \begin{bmatrix} p & 0 & 0 & q \\ q & p & 0 & 0 \\ 0 & 0 & p & q \\ 0 & 0 & q & p \end{bmatrix}$$

where $0 < p < 1$, $0 < q < 1$ and $p + q = 1$.

(Continued on page 2.)

a) Describe the Markov chain by a diagram.

SOLUTION: See Figure 1.

b) The chain has three classes, $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{2\}$ and $\mathcal{C}_3 = \{3, 4\}$. For each of these classes discuss whether the class is *transient* or *recurrent*.

SOLUTION: We consider the probabilities:

$$f_i = P\left(\bigcup_{r=1}^{\infty} \{X_r = i\} \mid X_0 = i\right), \quad i \in \mathcal{X}.$$

From the textbook we have that:

- State i is *recurrent* if $f_i = 1$.
- State i is *transient* if $f_i < 1$.

In this case we have:

$$f_1 = P\left(\bigcup_{r=1}^{\infty} \{X_r = 1\} \mid X_0 = 1\right) = 1 - P\left(\bigcap_{r=1}^{\infty} \{X_r \neq 1\} \mid X_0 = 1\right)$$
$$= 1 - P(X_1 = 4 \mid X_0 = 1) = 1 - q < 1.$$

$$f_2 = P\left(\bigcup_{r=1}^{\infty} \{X_r = 2\} \mid X_0 = 2\right) = 1 - P\left(\bigcap_{r=1}^{\infty} \{X_r \neq 2\} \mid X_0 = 2\right)$$
$$= 1 - P(X_1 = 1 \mid X_0 = 2) = 1 - q < 1.$$

$$f_3 = P\left(\bigcup_{r=1}^{\infty} \{X_r = 3\} \mid X_0 = 3\right) = 1 - P\left(\bigcap_{r=1}^{\infty} \{X_r = 4\} \mid X_0 = 3\right)$$
$$= 1 - \lim_{n \to \infty} qp^n = 1 - 0 = 1$$

$$f_4 = P\left(\bigcup_{r=1}^{\infty} \{X_r = 4\} \mid X_0 = 4\right) = 1 - P\left(\bigcap_{r=1}^{\infty} \{X_r = 3\} \mid X_0 = 4\right)$$
$$= 1 - \lim_{n \to \infty} qp^n = 1 - 0 = 1$$

Hence, we conclude that $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{2\}$ are transient, while $\mathcal{C}_3 = \{3, 4\}$ is recurrent.

c) Show that the two-step transition probability matrix is given by:

$$\boldsymbol{P}^{(2)} = \begin{bmatrix} p^2 & 0 & q^2 & 2pq \\ 2pq & p^2 & 0 & q^2 \\ 0 & 0 & p^2 + q^2 & 2pq \\ 0 & 0 & 2pq & p^2 + q^2 \end{bmatrix}$$

(Continued on page 3.)

SOLUTION: We have that:

$$\boldsymbol{P}^{(2)} = \boldsymbol{P} \cdot \boldsymbol{P} = \begin{bmatrix} p & 0 & 0 & q \\ q & p & 0 & 0 \\ 0 & 0 & p & q \\ 0 & 0 & q & p \end{bmatrix} \cdot \begin{bmatrix} p & 0 & 0 & q \\ q & p & 0 & 0 \\ 0 & 0 & p & q \\ 0 & 0 & q & p \end{bmatrix}$$

$$= \begin{bmatrix} p^2 & 0 & q^2 & 2pq \\ 2pq & p^2 & 0 & q^2 \\ 0 & 0 & p^2+q^2 & 2pq \\ 0 & 0 & 2pq & p^2+q^2 \end{bmatrix}$$

In more detail:

$$P_{ij}^2 = \sum_{k \in \mathcal{X}} P_{ik} \cdot P_{kj}, \quad \text{for all } i, j \in \mathcal{X}.$$

Hence, we have:

$$P_{1,1}^2 = P_{1,1}P_{1,1} + P_{1,2}P_{2,1} + \cdots + P_{1,4}P_{4,1} = p^2$$
$$P_{1,2}^2 = P_{1,1}P_{1,2} + P_{1,2}P_{2,2} + \cdots + P_{1,4}P_{4,2} = 0$$
$$P_{1,3}^2 = P_{1,1}P_{1,3} + P_{1,2}P_{2,3} + \cdots + P_{1,4}P_{4,3} = q^2$$
$$P_{1,4}^2 = P_{1,1}P_{1,4} + P_{1,2}P_{2,4} + \cdots + P_{1,4}P_{4,4} = 2pq$$

......

d) Conditioned upon the chain has entered one of the states 3 or 4 find the stationary distribution over these two states.

SOLUTION: We let:

$$\boldsymbol{Q} = \begin{bmatrix} P_{3,3} & P_{3,4} \\ P_{4,3} & P_{4,4} \end{bmatrix} = \begin{bmatrix} p & q \\ q & p \end{bmatrix}$$

denote the submatrix of $\boldsymbol{Q}$ containing the transition probabilities for the recurrent states 3 and 4. Furthermore, we let $\boldsymbol{\pi} = (\pi_3, \pi_4)$ denote the stationary distribution over these states. Then $\boldsymbol{\pi}$ must satisfy $\pi_3 + \pi_4 = 1$ and:

$$\boldsymbol{\pi Q} = \boldsymbol{\pi}$$

From the last set of equations we get that:

$$p\pi_3 + q\pi_4 = \pi_3$$

(Continued on page 4.)

or equivalently, since $q = 1 - p$ that:

$$q\pi_4 = (1-p)\pi_3 = q\pi_3$$

Hence, we conclude that $\pi_3 = \pi_4$, and since $\pi_3 + \pi_4 = 1$ it follows that:

$$\pi_3 = \pi_4 = \tfrac{1}{2}$$

e) We assume that $X_0 = 1$, and let M be given by:

$$M = \min\{m > 0 : X_m = 4\}$$

Thus, M is the number of steps until the Markov chain enters state 4 for the first time given that the chain starts out in state 1. Show that the probability distribution of M is given by:

$$P(M = m) = p^{m-1}q, \quad m = 1, 2, \ldots$$

SOLUTION: The result follows since in this case:

$$\begin{aligned} P(M = m) &= P(X_1 = 1, \ldots, X_{m-1} = 1, X_m = 4 | X_0 = 1) \\ &= [P_{1,1}]^{m-1} \cdot P_{1,4} = p^{m-1}q, \quad m = 1, 2, \ldots \end{aligned}$$

f) Find $E[M]$.

SOLUTION: We observe that M has a geometric distribution. Hence, it follows that:

$$E[M] = \frac{1}{q}.$$

g) In the remaining part of this problem we assume that $X_0 = 2$, and let N be given by:

$$N = \min\{n > 0 : X_n = 3\}$$

Thus, N is the number of steps until the Markov chain enters state 3 for the first time given that the chain starts out in state 2. Find $E[N]$.

SOLUTION: We note that if $X_0 = 2$ and n is the number of steps until the Markov chain enters state 3 for the first time, the chain must have gone through the states 1 and 4 before entering 3. Thus, we introduce the following stochastic variables:

$$N_1 = \min\{n > 0 : X_n = 1\}$$

$$N_2 = \min\{n > 0 : X_{N_1+n} = 4\}$$

$$N_3 = \min\{n > 0 : X_{N_2+n} = 3\}$$

(Continued on page 5.)

Using the same arguments as we used in (e) we get that:

$$P(N_i = n) = p^{n-1}q, \quad n = 1, 2, \dots \quad i = 1, 2, 3,$$

Moreover, we have that $N = N_1 + N_2 + N_3$, and so:

$$E[N] = E[N_1] + E[N_2] + E[N_3] = \frac{1}{q} + \frac{1}{q} + \frac{1}{q} = \frac{3}{q}.$$

h) Find the probability distribution of N.

SOLUTION: By the Markov property, it follows that N_1, N_2 and N_3 are independent random variables. Hence, N is a sum of three independent and geometrically distributed variables. By using the formula for the *negative binomial distribution* it follows that N has the following distribution:

$$P(N = n) = \binom{n-1}{2} p^{n-3} q^3, \quad n = 3, 4, 5 \dots$$

Alternatively, we may argue in more details, and introduce:

$$J_i = I(X_i \neq X_{i-1}), \quad i = 1, 2, \dots$$

Thus, J_i is *one* if and only if the Markov chain *changes its state* at the ith step and zero otherwise. For the given transition probability matrix we have:

$$P(J_i = 1) = q, \quad i = 1, 2, \dots$$

Moreover, by the Markov property, it follows that $J_1, J_2, \dots$ are independent. Hence, $(J_1 + \cdots + J_k) \sim Bin(k, q)$, $k = 1, 2, \dots$. We also note that $(J_1 + \cdots + J_k)$ is equal to the *number of state changes* in the first k steps.

The event that $N = n$ is equivalent to the event that there are exactly 2 state changes (one from state 2 to state 1, and one from state 1 to state 4) among the first $n - 1$ steps, and that the third state change (from state 4 to state 3) happens at the nth step. From this it follows that:

$$\begin{aligned} P(N = n) &= P(J_1 + \cdots + J_{n-1} = 2) \cdot P(J_n = 1) \\ &= \binom{n-1}{2} p^{(n-1)-2} q^2 \cdot q \\ &= \binom{n-1}{2} p^{n-3} q^3, \quad n = 3, 4, \dots \end{aligned}$$

(Continued on page 6.)

Problem 2

An urn always contains 2 balls. The balls are colored either *red* or *blue*. At each stage a ball is randomly chosen and then replaced by a new ball, which with probability 0.75 is the *same color*, and with probability 0.25 is the *opposite color*, as the ball it replaces. This is modelled by a Markov chain $\{X_n : n \geq 0\}$ where:

$X_n =$ The number of *red* balls after the nth selection. $n = 0, 1, 2, \ldots$

Thus, the state space of the Markov chain is $\mathcal{X} = \{0, 1, 2\}$.

a) Explain why the transition probability matrix of this Markov chain is:

$$\boldsymbol{P} = \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.125 & 0.75 & 0.125 \\ 0 & 0.25 & 0.75 \end{bmatrix}$$

SOLUTION: The transition probabilities are:

$P_{00} = P\{\text{Blue ball selected but not replaced}\} = 1.0 \cdot 0.75 = 0.75,$
$P_{01} = P\{\text{Blue ball selected and replaced by red}\} = 1.0 \cdot 0.25 = 0.25,$
$P_{02} = 0.0$

$P_{10} = P\{\text{Red ball selected and replaced by blue}\} = 0.5 \cdot 0.25 = 0.125$
$P_{11} = P\{\text{Any ball selected but not replaced}\} = 0.75$
$P_{12} = P\{\text{Blue ball selected and replaced by red}\} = 0.5 \cdot 0.25 = 0.125$

$P_{20} = 0.0$
$P_{21} = P\{\text{Red ball selected and replaced by blue}\} = 1.0 \cdot 0.25 = 0.25$
$P_{22} = P\{\text{Red ball selected but not replaced}\} = 1.0 \cdot 0.75 = 0.75$

b) It can be calculated that:

$$\boldsymbol{P}^{(4)} \approx \begin{bmatrix} 0.4238 & 0.4688 & 0.1074 \\ 0.2344 & 0.5313 & 0.2344 \\ 0.1074 & 0.4688 & 0.4238 \end{bmatrix}$$

You do not need to calculate this.

Find the probability that the fifth ball selected is *red* given that $X_0 = 2$.

SOLUTION:

$$\begin{aligned} & P(\text{Selection 5 is red}) \\ &= \sum_{i=0}^{2} P(\text{Selection 5 is red} | X_4 = i) \cdot P(X_4 = i | X_0 = 2) \\ &= 0.00 \cdot P_{2,0}^4 + 0.50 \cdot P_{2,1}^4 + 1.00 \cdot P_{2,2}^4 \\ &= 0.50 \cdot 0.4688 + 0.4238 = 0.6582 \end{aligned}$$

(Continued on page 7.)

c) Find the stationary distribution for the Markov chain $\{X_n : n \geq 0\}$.

SOLUTION: We let $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2)$ denote the stationary distribution over $\mathcal{X} = \{0, 1, 2\}$. Then $\boldsymbol{\pi}$ must satisfy $\pi_0 + \pi_1 + \pi_2 = 1$ and:

$$\boldsymbol{\pi P} = \boldsymbol{\pi}$$

Using the first and the last equation, we get:

$$\begin{aligned} \tfrac{3}{4}\pi_0 + \tfrac{1}{8}\pi_1 &= \pi_0 \\ \tfrac{3}{4}\pi_2 + \tfrac{1}{8}\pi_1 &= \pi_2 \end{aligned}$$

This is equivalent to:

$$\begin{aligned} \pi_0 &= \tfrac{1}{2}\pi_1 \\ \pi_2 &= \tfrac{1}{2}\pi_1 \end{aligned}$$

Inserting this into the equation $\pi_0 + \pi_1 + \pi_2 = 1$ we get:

$$\tfrac{1}{2}\pi_1 + \pi_1 + \tfrac{1}{2}\pi_1 = 2\pi_1 = 1$$

From this we get the stationary distribution $\pi_0 = \frac{1}{4}$, $\pi_1 = \frac{1}{2}$, $\pi_2 = \frac{1}{4}$.

d) Let ρ_n denote the probability that the nth ball selected is *red* given that $X_0 = 2$. Find:

$$\lim_{n\to\infty} \rho_n$$

SOLUTION:

$$\begin{aligned} \rho_n &= P(\text{The } n\text{th selection is red}) \\ &= \sum_{i=0}^{2} P(\text{The } n\text{th selection is red}|X_{n-1} = i) \cdot P(X_{n-1} = i|X_0 = 2) \\ &= 0.00 \cdot P_{2,0}^{n-1} + 0.50 \cdot P_{2,1}^{n-1} + 1.00 \cdot P_{2,2}^{n-1} \\ &= 0.50 \cdot P_{2,1}^{n-1} + 1.00 \cdot P_{2,2}^{n-1} \end{aligned}$$

Hence, we get:

$$\begin{aligned} \lim_{n\to\infty} \rho_n &= 0.50 \cdot \lim_{n\to\infty} P_{2,1}^{n-1} + 1.00 \cdot \lim_{n\to\infty} P_{2,2}^{n-1} \\ &= 0.50 \cdot \pi_1 + 1.00 \cdot \pi_2 = 0.50 \cdot \tfrac{1}{2} + 1.00 \cdot \tfrac{1}{4} = \tfrac{1}{2}. \end{aligned}$$

(Continued on page 8.)

e) We now introduce:

$$N_j = \min\{n > 0 : X_n = j\}, \quad j \in \mathcal{X}.$$

Thus, N_j is the number of steps until the Markov chain makes a transition into state j. We then let:

$$m_j = E[N_j | X_0 = j], \quad j \in \mathcal{X}.$$

That is, m_j is the expected number of steps until the Markov chain returns to state j given that it starts out in state j.

Find m_j for all $j \in \mathcal{X}$.

SOLUTION: In the textbook (Ross (2019) page 216) we have the following result:

PROPOSITION 4.4. If a Markov chain is irreducible and recurrent, then for any initial state X_0, we have:

$$\pi_j = 1/m_j, \qquad \text{for all } j \in \mathcal{X}.$$

Since the Markov chain under consideration is irreducible and recurrent, it follows that:

$$\begin{aligned} m_0 &= 1/\pi_0 = 1/0.25 = 4 \\ m_1 &= 1/\pi_1 = 1/0.50 = 2 \\ m_2 &= 1/\pi_2 = 1/0.25 = 4 \end{aligned}$$

Problem 3

In this problem we consider a population consisting of individuals able to produce offspring of the same kind. We assume that each individual will, by the end of its lifetime, have produced r new offspring with probability p_r, $r = 0, 1, 2, \ldots$, independently of the numbers produced by other individuals. We assume that $p_0 > 0$, and that $p_r < 1$ for $r = 0, 1, 2, \ldots$.

The number of individuals initially present, denoted by X_0, is called the size of the 0-th generation. Moreover, we let:

$$X_n = \text{The population size in the } n\text{th generation}, \quad n = 0, 1, 2, \ldots$$

a) Explain why $\{X_n : n \geq 0\}$ is a *Markov chain*.

SOLUTION: Given the population size in the nth generation, i.e., X_n, the number of offspring produced by this generation, i.e., X_{n+1} depends only on X_n and not on the size of the previous generations. This follows since X_{n+1} given $X_n = x_n$ is the sum of x_n independent and identically distributed variables. Hence, $\{X_n : n \geq 0\}$ is a *Markov chain*.

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b) Explain why state 0 is a *recurrent state*, and why any state $j > 0$ is *transient*.

SOLUTION: Since $P_{00} = 1$, then 0 is a *recurrent state*.

Since $p_0 > 0$, it follows that $P_{j0} = p_0^j > 0$. Hence, any state $j > 0$ is *transient*.

c) In the rest of this problem we assume that $X_0 = 1$, and $E[X_1] = \mu$. Show that $E[X_n] = \mu^n$.

SOLUTION: We start out by introducing for $r = 1, \ldots, X_{n-1}$:

$Z_r =$ Number of offspring from individual r in generation $(n-1)$.

By conditioning on X_{n-1} we get:

$$
\begin{aligned}
E[X_n] &= E[E[X_n \mid X_{n-1}]] \\
&= E[E[\sum_{r=1}^{X_{n-1}} Z_r \mid X_{n-1}]] \\
&= E[X_{n-1}\mu] = \mu E[X_{n-1}]
\end{aligned}
$$

Since we have assumed that $X_0 = 1$, it follows by induction that:

$$E[X_n] = \mu^n.$$

d) We then consider the probability that the population eventually dies out:

$$\pi_0 = \lim_{n\to\infty} P(X_n = 0 \mid X_0 = 1)$$

Show that π_0 satisfies the following equation:

$$\pi_0 = \sum_{r=0}^{\infty} \pi_0^r \, p_r \tag{1}$$

SOLUTION: By conditioning on X_1 we get:

$$
\begin{aligned}
\pi_0 &= P(\text{The population dies out}) \\
&= \sum_{j=0}^{\infty} P(\text{The population dies out} \mid X_1 = j) p_j \\
&= \sum_{j=0}^{\infty} \pi_0^j p_j
\end{aligned}
$$

(Continued on page 10.)

e) In the following you may use without proof that π_0 is the *smallest positive number* that satisfies (1).

Assume that $p_0 = \frac{1}{5}$, $p_1 = \frac{1}{5}$, $p_2 = \frac{3}{5}$, and that $p_r = 0$, for $r > 2$.

Calculate μ and π_0 in this case.

SOLUTION: If $p_0 = \frac{1}{5}$, $p_1 = \frac{1}{5}$, $p_2 = \frac{3}{5}$, and that $p_r = 0$, for $r > 2$, we get:

$$\begin{aligned}\mu = E[X_1] &= \sum_{r=1}^{\infty} r \cdot p_r \\ &= 1 \cdot p_1 + 2 \cdot p_2 = 1 \cdot \tfrac{1}{5} + 2 \cdot \tfrac{3}{5} \\ &= \tfrac{1}{5} + \tfrac{6}{5} = \tfrac{7}{5}.\end{aligned}$$

In order to determine π_0, we use the fact that π_0 is the *smallest positive number* that satisfies (1). In this case (1) is reduced to:

$$\begin{aligned}\pi_0 &= \pi_0^0 p_0 + \pi_0^1 p_1 + \pi_0^2 p_2 \\ &= \tfrac{1}{5} + \tfrac{1}{5}\pi_0 + \tfrac{3}{5}\pi_0^2\end{aligned}$$

This is equivalent to:

$$3\pi_0^2 - 4\pi_0 + 1 = 0$$

which has the two solutions: $\pi_0 = \frac{1}{3}$ and $\pi_0 = 1$. Since π_0 is the *smallest positive number* that satisfies (1), we get that:

$$\pi_0 = \tfrac{1}{3}.$$

Problem 4

A system can be in three possible states denoted respectively 0, 1 and 2. If the system is in state 0, it is considered to be *failed*, while if the system is in state 2, it is considered to be *functioning perfectly*. The state 1 represents an intermediate case where the system is functioning, but at a lower performance level than when it is in state 2.

We model this as a continuous-time Markov chain $\{X(t) : t \geq 0\}$ with state space $\mathcal{X} = \{0, 1, 2\}$. The system can transit from state i to state $i+1$ with rate μ, $i = 0, 1$. Such a transition is called a *repair*. Moreover, the system can transit from state i to state $i-1$ with rate λ, $i = 1, 2$. Such a transition is called a *failure*. Thus, a single repair can only increase the state by 1. Similarly, a single failure can only reduce the state by 1. It is *not possible* to transit directly from state 0 to state 2 or directly from state 2 to state 0. Finally, we assume that $\mu > 0$ and $\lambda > 0$.

(Continued on page 11.)

We also introduce:

$$P_{ij}(t) = P(X(t) = j | X(0) = i), \quad \text{for all } i, j \in \mathcal{X}.$$

Moreover, for all $i, j \in \mathcal{X}$ we let:

$$q_{ij} = \text{The transition rate from state } i \text{ to state } j \text{ if } i \neq j.$$

$$v_i = \sum_{j \in \mathcal{X} \setminus i} q_{ij}.$$

Finally, we let the matrix $\boldsymbol{R}$ be given by:

$$\boldsymbol{R} = \begin{bmatrix} -v_0 & q_{0,1} & q_{0,2} \\ q_{1,0} & -v_1 & q_{1,2} \\ q_{2,0} & q_{2,1} & -v_2 \end{bmatrix}$$

a) Determine the matrix $\boldsymbol{R}$ expressed in terms of μ and λ.

SOLUTION: It follows that:

$$q_{i,i+1} = \mu, \quad i = 0, 1$$

$$q_{i,i-1} = \lambda, \quad i = 1, 2$$

$$q_{i,j} = 0, \quad \text{for all } i, j \in \mathcal{X} \text{ such that } |i - j| > 1.$$

From this we also get that:

$$v_0 = q_{0,1} + q_{0,2} = \mu + 0 = \mu$$

$$v_1 = q_{1,0} + q_{1,2} = \lambda + \mu$$

$$v_2 = q_{2,0} + q_{2,1} = 0 + \lambda = \lambda$$

Hence, the matrix $\boldsymbol{R}$ is given by:

$$\boldsymbol{R} = \begin{bmatrix} -\mu & \mu & 0 \\ \lambda & -(\lambda + \mu) & \mu \\ 0 & \lambda & -\lambda \end{bmatrix}$$

b) Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2)$, where:

$$\pi_j = \lim_{t \to \infty} P_{ij}(t), \quad \text{for all } j \in \mathcal{X}.$$

Formulate a set of equations which can be used to determine $\boldsymbol{\pi}$, and solve these equations.

SOLUTION: We know that $\pi_0 + \pi_1 + \pi_2 = 1$. Furthermore, by using Kolmogorov's forward equations, and taking the limit as t goes to infinity, we get the following set of equations:

$$\boldsymbol{\pi R} = 0$$

(Continued on page 12.)

Using the first and the last equation, we get:

$$\mu\pi_0 = \lambda\pi_1$$

$$\mu\pi_1 = \lambda\pi_2$$

We then use these equations to express π_1 and π_2 in terms of π_0:

$$\pi_1 = \frac{\mu}{\lambda}\pi_0$$

$$\pi_2 = \frac{\mu}{\lambda}\pi_1 = \frac{\mu^2}{\lambda^2}\pi_0$$

Since $\pi_0 + \pi_1 + \pi_2 = 1$, we get:

$$\pi_0 + \pi_1 + \pi_2 \;=\; \pi_0 + \frac{\mu}{\lambda}\pi_0 + \frac{\mu^2}{\lambda^2}\pi_0 \;=\; \pi_0\left(1 + \frac{\mu}{\lambda} + \frac{\mu^2}{\lambda^2}\right) \;=\; 1$$

Hence, we obtain the following solution:

$$\pi_0 = \frac{1}{1 + \frac{\mu}{\lambda} + \frac{\mu^2}{\lambda^2}}, \qquad \pi_1 = \frac{\frac{\mu}{\lambda}}{1 + \frac{\mu}{\lambda} + \frac{\mu^2}{\lambda^2}}, \qquad \pi_2 = \frac{\frac{\mu^2}{\lambda^2}}{1 + \frac{\mu}{\lambda} + \frac{\mu^2}{\lambda^2}}$$

or alternatively:

$$\pi_0 = \frac{\lambda^2}{\lambda^2 + \lambda\mu + \mu^2}, \qquad \pi_1 = \frac{\lambda\mu}{\lambda^2 + \lambda\mu + \mu^2}, \qquad \pi_2 = \frac{\mu^2}{\lambda^2 + \lambda\mu + \mu^2}$$

c) Assume that $X(0) = 2$, and let:

$$T = \min\{t > 0 : X(t) \neq 2\}$$

Explain briefly why we have:

$$P(T > t) = e^{-\lambda t}, \quad \text{for all } t > 0.$$

SOLUTION: Since $\{X(t) : t \geq 0\}$ is a continuous-time Markov chain, we know that the times between transitions are independent and *exponentially distributed*. Since T is the time to the first transition, and the only possible transition from state 2 is a transition to state 1, which happens at rate λ, it follows that $T \sim exp(q_{2,1}) = exp(\lambda)$. Hence, we have:

$$P(T > t) = \int_t^\infty \lambda e^{-\lambda u} du = e^{-\lambda t}, \quad \text{for all } t > 0.$$

(Continued on page 13.)

d) We still assume that $X(0) = 2$. However, we now consider the case where $\mu = 0$, and let:

$$S = \min\{t > 0 : X(t) = 0\}$$

What is the probability distribution of S? Explain your answer.

SOLUTION: When $\mu = 0$, the only transitions possible are transitions from state 2 to state 1 and transitions from state 1 to state 0.

At time S we know that the Markov chain has had exactly two transitions: one from state 2 to state 1 and one from state 1 to state 0. We denote the times between these transitions by T_1 and T_2 respectively. By the Markov property it follows that T_1 and T_2 are *independent*.

By the same arguments we used in the previous point, it follows that $T_i \sim exp(\lambda)$, $i = 1, 2$.

Finally, since $S = T_1 + T_2$, it follows that $S \sim Gamma(2, \lambda)$. Thus, the density of S is given by:

$$\begin{aligned} f_S(s) &= \frac{\lambda^2}{\Gamma(2)} s^{2-1} e^{-\lambda s} \\ &= \lambda^2 s e^{-\lambda s}, \quad s > 0. \end{aligned}$$

Problem 5

Let $\{X(t) : t \geq 0\}$ be a standard Brownian motion process, and let $0 < t_1 < t_2$.

a) Find the joint density of $X_1 = X(t_1)$ and $X_2 = X(t_2)$.

SOLUTION: We want to determine the joint density of X_1 and X_2, which denote by $f_{\boldsymbol{t}}(x_1, x_2)$, where $\boldsymbol{t} = (t_1, t_2)$.
In order to do so, we let $Y_1 = X_1$ and $Y_2 = X_2 - X_1$. By the properties of a standard Brownian motion process it follows that Y_1 and Y_2 are independent, and that $Y_1 \sim N(0, t_1)$ and $Y_2 \sim N(0, t_2 - t_1)$. Hence, the joint density of Y_1 and Y_2 is given by:

$$f_{t_1}(y_1) \cdot f_{t_2 - t_1}(y_2)$$

where:

$$f_t(y) = \frac{1}{\sqrt{2\pi t}} e^{-y^2/(2t)}, \quad t > 0, \quad -\infty < y < \infty.$$

The joint density of X_1 and X_2 is then obtained by transforming the Y_is to the X_is.

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This is a simple *linear transformation* with a Jacobian given by:

$$J = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

The *Jacobian determinant* of this transformation is 1. Thus, by the change of variable formula we get that:

$$f_{\boldsymbol{t}}(x_1, x_2) = f_{t_1}(x_1) \cdot f_{t_2 - t_1}(x_2 - x_1)$$

More specifically, the joint density has the form:

$$f_{\boldsymbol{t}}(x_1, x_2) = C(\boldsymbol{t}) e^{-(1/2)Q(x_1, x_2)}$$

where $C(\boldsymbol{t})$ is a suitable normalizing constant, and where:

$$Q(x_1, x_2) = \frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1}$$

b) Show that $(X_2 | X_1 = x_1) \sim N(x_1, t_2 - t_1)$.

SOLUTION: By the properties of a standard Brownian motion we know that $X_i \sim N(0, t_i)$, $i = 1, 2$. Hence, the marginal densities of X_1 and X_2 are respectively:

$$f_{t_1}(x_1) = C(t_1) e^{-(1/2)(x_1^2/t_1)}$$

$$f_{t_2}(x_2) = C(t_2) e^{-(1/2)(x_2^2/t_2)}$$

The conditional density of X_2 given $X_1 = x_1$ then becomes:

$$f_{X_2|X_1 = x_1} = \frac{f_{\boldsymbol{t}}(x_1, x_2)}{f_{t_1}(x_1)} = \frac{C(\boldsymbol{t}) e^{-(1/2)\left[\frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1}\right]}}{C(t_1) e^{-(1/2)\left[\frac{x_1^2}{t_1}\right]}}$$

$$= C(t_2 | t_1) e^{-(1/2)\left[\frac{(x_2 - x_1)^2}{t_2 - t_1}\right]}$$

where the normalizing constant $C(t_2 | t_1) = C(\boldsymbol{t})/C(t_1)$.

From this it follows that $(X_2 | X_1 = x_1) \sim N(x_1, t_2 - t_1)$.

c) Show that $(X_1 | X_2 = x_2) \sim N(\frac{t_1}{t_2} x_2, \frac{t_1}{t_2}(t_2 - t_1))$.

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SOLUTION: In order to find the conditional density of X_1 given $X_2 = x_2$, we rewrite $Q(x_1, x_2)$ as follows:

$$\begin{aligned}
Q(x_1, x_2) &= \frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} = \frac{x_1^2}{t_1} + \frac{x_2^2 - 2x_2x_1 + x_1^2}{t_2 - t_1} \\
&= \left[\frac{1}{t_1} + \frac{1}{t_2 - t_1}\right] x_1^2 - \frac{2x_2}{t_2 - t_1} x_1 + \frac{1}{t_2 - t_1} x_2^2 \\
&= \frac{t_2}{t_1(t_2 - t_1)} x_1^2 - \frac{2x_2}{t_2 - t_1} x_1 + \frac{1}{t_2 - t_1} x_2^2 \\
&= \frac{t_2}{t_1(t_2 - t_1)} \left[x_1^2 - 2\frac{t_1 x_2}{t_2} x_1 + \frac{t_1}{t_2} x_2^2\right] \\
&= \frac{t_2}{t_1(t_2 - t_1)} \left[x_1^2 - 2\frac{t_1 x_2}{t_2} x_1 + \frac{t_1^2}{t_2^2} x_2^2 + \left(\frac{t_1}{t_2} - \frac{t_1^2}{t_2^2}\right) x_2^2\right] \\
&= \frac{t_2}{t_1(t_2 - t_1)} \left(x_1 - \frac{t_1}{t_2} x_2\right)^2 + \frac{t_2}{t_1(t_2 - t_1)} \frac{t_1}{t_2} \left(1 - \frac{t_1}{t_2}\right) x_2^2 \\
&= \frac{(x_1 - t_1 x_2 / t_2)^2}{t_1(t_2 - t_1)/t_2} + \frac{x_2^2}{t_2}
\end{aligned}$$

The conditional density of X_1 given $X_2 = x_2$ then becomes:

$$\begin{aligned}
f_{X_1|X_2=x_2} &= \frac{f_{\boldsymbol{t}}(x_1, x_2)}{f_{t_2}(x_2)} = \frac{C(\boldsymbol{t}) e^{-(1/2)\left[\frac{(x_1 - t_1 x_2/t_2)^2}{t_1(t_2 - t_1)/t_2} + \frac{x_2^2}{t_2}\right]}}{C(t_2) e^{-(1/2)\left[\frac{x_2^2}{t_2}\right]}} \\
&= C(t_1|t_2) e^{-(1/2)\left[\frac{(x_1 - t_1 x_2/t_2)^2}{t_1(t_2 - t_1)/t_2}\right]}
\end{aligned}$$

where the normalizing constant $C(t_1|t_2) = C(\boldsymbol{t})/C(t_2)$.

From this it follows that $(X_1|X_2 = x_2) \sim N(\frac{t_1}{t_2} x_2, \frac{t_1}{t_2}(t_2 - t_1))$.

d) Find $P(\max_{0 \leq s \leq 4} X(s) \geq 2)$.

SOLUTION: We introduce the following random variable:

$$T = \inf\{t > 0 : X(t) = 2\} = \text{The first time the process hits 2},$$

and note that:

$$\max_{0 \leq s \leq 4} X(s) \geq 2 \quad \Leftrightarrow \quad T \leq 4$$

Thus, we have that:

$$P(\max_{0 \leq s \leq 4} X(s) \geq 2) = P(T \leq 4)$$

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In order to find $P(T \leq 4)$, we instead consider $P(X(4) \geq 2)$, and condition on the event $\{T \leq 4\}$:

$$P(X(4) \geq 2) = P(X(4) \geq 2 | T \leq 4) P(T \leq 4) \\ + P(X(4) \geq 2 | T > 4) P(T > 4)$$

By symmetry, it follows that:

$$P(X(4) \geq 2 | T \leq 4) = \tfrac{1}{2}$$

Moreover, we obviously have:

$$P(X(4) \geq 2 | T > 4) = 0$$

Hence, we have:

$$P(X(4) \geq 2) = \tfrac{1}{2} P(T \leq 4)$$

and since $X(4) \sim N(0, 4)$, we get:

$$P(T \leq 4) = 2 \cdot P(X(4) \geq 2) = 2 \cdot P(\frac{X(4)}{\sqrt{4}} \geq \frac{2}{\sqrt{4}}) = 2 \cdot \Phi(-1),$$

where Φ denotes the cumulative distribution function of the standard normal distribution. Hence, we conclude that:

$$P(\max_{0 \leq s \leq 4} X(s) \geq 2) = P(T \leq 4) = 2\Phi(-1) \approx 2 \cdot 0.15865 = 0.31730$$

END