UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK2130 — Modelling by stochastic processes

Day of examination: Friday June 4th 2021.

Examination hours: 15.00 – 19.00

This problem set consists of 4 pages.

Appendices: None.

Permitted aids: All available notes and books.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Consider a discrete-time Markov chain $\{X_n : n \geq 0\}$ with state space $\mathcal{X} = \{0, 1, 2, 3\}$, and transition probability matrix:

$$\boldsymbol{P} = \begin{bmatrix} p & q & 0 & 0 \\ 0 & 0 & p & q \\ 0 & q & 0 & p \\ 0 & p & q & 0 \end{bmatrix}$$

where $0 < p < 1$, $0 < q < 1$ and $p + q = 1$.

a) Describe the Markov chain by a diagram.

b) The chain has two classes, $\mathcal{C}_1 = \{0\}$ and $\mathcal{C}_2 = \{1, 2, 3\}$. For each of these classes discuss whether the class is *transient* or *recurrent*.

c) Show that the two-step transition probability matrix is given by:

$$\boldsymbol{P}^{(2)} = \begin{bmatrix} p^2 & pq & pq & q^2 \\ 0 & 2pq & q^2 & p^2 \\ 0 & p^2 & 2pq & q^2 \\ 0 & q^2 & p^2 & 2pq \end{bmatrix}$$

d) Conditioned upon that the chain has entered $\mathcal{C}_2$, find the stationary distribution over these three states.

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Problem 2

A Markov chain is said to be *periodic* if it can only return to a state in a multiple of $d > 1$ steps. The smallest such number, d, is called the *period* of the Markov chain. A Markov chain which is not periodic, is said to be *aperiodic*.

Consider the Markov chain $\{X_n : n \geq 0\}$ with state space $\mathcal{X} = \{1, 2, 3, 4, 5\}$, and transition probability matrix:

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

a) Determine the period of this Markov chain.

b) Assume that $\{X_n : n \geq 0\}$ is an irreducible Markov chain with a finite state space $\mathcal{X}$. Moreover, assume that for some state $i \in \mathcal{X}$ we have:

$$P_{ii} = P(X_{n+1} = i | X_n = i) > 0$$

Explain why this Markov chain is aperiodic.

Problem 3

Consider a continuous-time Markov chain $\{X(t) : t \geq 0\}$ with state space $\mathcal{X} = \{1, 2, 3\}$. The transition probability matrix of the built-in discrete time Markov chain is given by:

$$\boldsymbol{Q} = \begin{bmatrix} Q_{1,1} & Q_{1,2} & Q_{1,3} \\ Q_{2,1} & Q_{2,2} & Q_{2,3} \\ Q_{3,1} & Q_{3,2} & Q_{3,3} \end{bmatrix} = \begin{bmatrix} 0 & p & q \\ q & 0 & p \\ p & q & 0 \end{bmatrix}$$

where $0 < p < 1$, $0 < q < 1$ and $p + q = 1$.

The amount of time spent in state i is exponentially distributed with rate λ_i, $i = 1, 2, 3$, and we let:

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

For all $i, j \in \mathcal{X}$ we let:

$$q_{ij} = \lambda_i Q_{ij} = \text{The transition rate from state } i \text{ to state } j \text{ if } i \neq j.$$

Finally, we let the matrix $\boldsymbol{R}$ be given by:

$$\boldsymbol{R} = \begin{bmatrix} -\lambda_1 & q_{1,2} & q_{1,3} \\ q_{2,1} & -\lambda_2 & q_{2,3} \\ q_{3,1} & q_{3,2} & -\lambda_3 \end{bmatrix}$$

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a) Show that:

$$\boldsymbol{R} = \boldsymbol{\Lambda}(\boldsymbol{Q} - \boldsymbol{I}),$$

where:

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) Assume that $\boldsymbol{\rho} = (\rho_1, \rho_2, \rho_3)$ is a vector such that:

$$\boldsymbol{\rho}\boldsymbol{Q} = \boldsymbol{\rho}$$

and let $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \kappa_3) = \boldsymbol{\rho}\boldsymbol{\Lambda}^{-1}$. Show that:

$$\boldsymbol{\kappa}\boldsymbol{R} = \boldsymbol{0}$$

c) We now introduce:

$$P_{ij}(t) = P(X(t) = j | X(0) = i), \quad \text{for all } i, j \in \mathcal{X},$$

and let $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$, where:

$$\pi_j = \lim_{t \to \infty} P_{ij}(t), \quad \text{for all } j \in \mathcal{X},$$

assuming that the limits exist.

Kolmogorov's forward equations can be written as:

$$\boldsymbol{P}'(t) = \boldsymbol{P}(t)\boldsymbol{R},$$

where:

$$\boldsymbol{P}(t) = \begin{bmatrix} P_{1,1}(t) & P_{1,2}(t) & P_{1,3}(t) \\ P_{2,1}(t) & P_{2,2}(t) & P_{2,3}(t) \\ P_{3,1}(t) & P_{3,2}(t) & P_{3,3}(t) \end{bmatrix}$$

Use this to show that $\boldsymbol{\pi}$ must satisfy the following set of equations:

$$\boldsymbol{\pi}\boldsymbol{R} = \boldsymbol{0}$$

d) Show that:

$$\pi_j = \frac{\lambda_j^{-1}}{\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1}}, \quad j = 1, 2, 3.$$

[Hint: Substitute $y_j = \lambda_j \pi_j$, $j = 1, 2, 3$ in the equations.]

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Problem 4

Let $\{N(t) : t \geq 0\}$ be a *renewal process* with *interarrival times* $X_1, X_2, \ldots$. The *renewal times*, denoted by $S_0, S_1, \ldots$, are given by:

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i, \quad n = 1, 2, \ldots$$

The cumulative distribution function of the interarrival times is denoted by F, and we let $\bar{F}(t) = 1 - F(t)$.

a) Show that:

$$P(N(t) = n) = \int_0^t \bar{F}(t-s) f_{S_n}(s) ds, \quad n = 1, 2, \ldots$$

where f_{S_n} denotes the density function of $S_n,\ n = 1, 2, \ldots$

b) Assume that $X_1, X_2, \ldots$ are independent and exponentially distributed with rate λ. Explain briefly why this implies that:

$$f_{S_n}(s) = \frac{\lambda^n}{(n-1)!} s^{n-1} e^{-\lambda s}, \quad s > 0, \quad n = 1, 2, \ldots$$

and use this to show that:

$$P(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 0, 1, \ldots$$

END