UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK2130 — Modelling by stochastic processes

Day of examination: Friday June 4th 2021.

Examination hours: 15.00 – 19.00

This problem set consists of 11 pages.

Appendices: None.

Permitted aids: All available notes and books.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

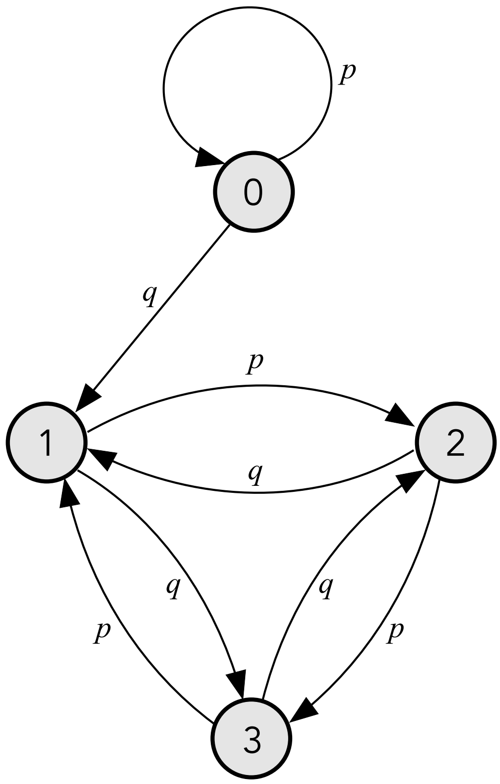

Figure 1: Diagram representing the Markov chain in Problem 1a

(Continued on page 2.)

Consider a discrete-time Markov chain $\{X_n : n \geq 0\}$ with state space $\mathcal{X} = \{0, 1, 2, 3\}$, and transition probability matrix:

$$\boldsymbol{P} = \begin{bmatrix} p & q & 0 & 0 \\ 0 & 0 & p & q \\ 0 & q & 0 & p \\ 0 & p & q & 0 \end{bmatrix}$$

where $0 < p < 1$, $0 < q < 1$ and $p + q = 1$.

a) Describe the Markov chain by a diagram.

SOLUTION:

See Figure 1.

b) The chain has two classes, $\mathcal{C}_1 = \{0\}$ and $\mathcal{C}_2 = \{1, 2, 3\}$. For each of these classes discuss whether the class is *transient* or *recurrent*.

SOLUTION:

We consider the probabilities:

$$f_i = P\left(\bigcup_{r=1}^{\infty} \{X_r = i\} \mid X_0 = i\right), \quad i \in \mathcal{X}.$$

From the textbook we have that state i is *transient* if $f_i < 1$ and *recurrent* if $f_i = 1$.

In this case we have:

$$f_0 = P\left(\bigcup_{r=1}^{\infty} \{X_r = 0\} \mid X_0 = 0\right) = 1 - P\left(\bigcap_{r=1}^{\infty} \{X_r \neq 0\} \mid X_0 = 0\right)$$
$$= 1 - P(X_1 = 1 \mid X_0 = 0) = 1 - q < 1.$$

Hence, we conclude that $\mathcal{C}_1 = \{0\}$ is *transient*.

From the diagram it is easy to see that $i \leftrightarrow j$ for all $i, j \in \mathcal{C}_2$. Hence, these states belong to the same class. Since transience and recurrence are class properties, it follows that either all states in $\mathcal{C}_2$ are *transient* or all states in $\mathcal{C}_2$ are *recurrent*. However, the Markov chain has a finite state space, which implies that at least one state must be *recurrent*. Thus, the only possibility is that $\mathcal{C}_2 = \{1, 2, 3\}$ is *recurrent*.

c) Show that the two-step transition probability matrix is given by:

$$\boldsymbol{P}^{(2)} = \begin{bmatrix} p^2 & pq & pq & q^2 \\ 0 & 2pq & q^2 & p^2 \\ 0 & p^2 & 2pq & q^2 \\ 0 & q^2 & p^2 & 2pq \end{bmatrix}$$

(Continued on page 3.)

SOLUTION:

We have that:

$$\boldsymbol{P}^{(2)} = \boldsymbol{P} \cdot \boldsymbol{P} = \begin{bmatrix} p & q & 0 & 0 \\ 0 & 0 & p & q \\ 0 & q & 0 & p \\ 0 & p & q & 0 \end{bmatrix} \cdot \begin{bmatrix} p & q & 0 & 0 \\ 0 & 0 & p & q \\ 0 & q & 0 & p \\ 0 & p & q & 0 \end{bmatrix}$$

$$= \begin{bmatrix} p^2 & pq & pq & q^2 \\ 0 & 2pq & q^2 & p^2 \\ 0 & p^2 & 2pq & q^2 \\ 0 & q^2 & p^2 & 2pq \end{bmatrix}$$

In more detail:

$$P_{ij}^2 = \sum_{k \in \mathcal{X}} P_{ik} \cdot P_{kj}, \qquad \text{for all } i, j \in \mathcal{X}.$$

Hence, we have:

$$P_{0,0}^2 = P_{0,0}P_{0,0} + P_{0,1}P_{1,0} + \cdots + P_{0,3}P_{3,0} = p^2$$

$$P_{0,1}^2 = P_{0,0}P_{0,1} + P_{0,1}P_{1,1} + \cdots + P_{0,3}P_{3,1} = pq$$

$$P_{0,2}^2 = P_{0,0}P_{0,2} + P_{0,1}P_{1,2} + \cdots + P_{0,3}P_{3,2} = pq$$

$$P_{0,3}^2 = P_{0,0}P_{0,3} + P_{0,1}P_{1,3} + \cdots + P_{0,3}P_{3,4} = q^2$$

$$\cdots\cdots$$

d) Conditioned upon that the chain has entered $\mathcal{C}_2$, find the stationary distribution over these three states.

SOLUTION:

We let:

$$\boldsymbol{Q} = \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,1} & P_{3,2} & P_{3,3} \end{bmatrix} = \begin{bmatrix} 0 & p & q \\ q & 0 & p \\ p & q & 0 \end{bmatrix}$$

denote the submatrix of $\boldsymbol{P}$ containing the transition probabilities for the recurrent states $1, 2, 3$. Furthermore, we let $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ denote the stationary distribution over these states. Then $\boldsymbol{\pi}$ must satisfy $\pi_1 + \pi_2 + \pi_3 = 1$ and:

$$\boldsymbol{\pi}\boldsymbol{Q} = \boldsymbol{\pi}$$

(Continued on page 4.)

From the last set of equations we get that:

$$q\pi_2 + p\pi_3 = \pi_1$$
$$p\pi_1 + q\pi_3 = \pi_2$$

Since $q = 1 - p$, these equations can be written as:

$$(1-p)\pi_2 + p\pi_3 = \pi_1$$
$$p\pi_1 + (1-p)\pi_3 = \pi_2$$

We then multiply the first equation by p, and rearrange the terms:

$$p^2\pi_3 = p\pi_1 - p(1-p)\pi_2$$
$$(1-p)\pi_3 = -p\pi_1 + \pi_2$$

We then add the two equations and get:

$$(p^2 - p + 1)\pi_3 = (p^2 - p + 1)\pi_2$$

This implies that $\pi_2 = \pi_3$. By inserting this into e.g., the first equation, we get that:

$$q\pi_2 + p\pi_2 = \pi_1$$

This implies that $\pi_2 = \pi_1$. Thus, we conclude that $\pi_1 = \pi_2 = \pi_3$, and since also $\pi_1 + \pi_2 + \pi_3 = 1$, it follows that:

$$\pi_1 = \pi_2 = \pi_3 = \tfrac{1}{3}$$

Alternatively, the result that the stationary distribution is *uniform* follows directly by the fact that the matrix $\boldsymbol{Q}$ is *doubly stochastic.*

Problem 2

A Markov chain is said to be *periodic* if it can only return to a state in a multiple of $d > 1$ steps. The smallest such number, d, is called the *period* of the Markov chain. A Markov chain which is not periodic, is said to be *aperiodic.*

Consider the Markov chain $\{X_n : n \geq 0\}$ with state space $\mathcal{X} = \{1, 2, 3, 4, 5\}$, and transition probability matrix:

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(Continued on page 5.)

a) Determine the period of this Markov chain.

SOLUTION:

We start out by noting that in the given Markov chain we have $i \leftrightarrow j$ for all $i, j \in \mathcal{X}$. Thus, the Markov chain is irreducible, i.e., all states belong to the same class. The periodicity of an irreducible Markov chain is a class property. Thus, in order to determine the period, we may choose any state $i \in \mathcal{X}$, and consider:

$$P_{ii}^n = \text{The probability that the chain returns to state } i \text{ in } n \text{ steps}$$

If $P_{ii}^n > 0$ if and only if n is a multiple of d, then the period of the chain is d.

In this case we let $i = 3$, and note that there are exactly two paths from state 3 back to state 3:

$$3 \to 1 \to 2 \to 3$$
$$3 \to 5 \to 4 \to 3$$

Since both paths have length 3, it follows that:

$$P_{3,3}^n > 0 \quad \text{if and only if} \quad n = 3, 6, 9, \dots$$

Thus, we conclude that the period of the Markov chain is 3.

b) Assume that $\{X_n : n \geq 0\}$ is an irreducible Markov chain with a finite state space $\mathcal{X}$. Moreover, assume that for some state $i \in \mathcal{X}$ we have:

$$P_{ii} = P(X_{n+1} = i | X_n = i) > 0$$

Explain why this Markov chain is aperiodic.

SOLUTION:

In this case it follows that:

$$P_{ii}^n > (P_{ii})^n > 0 \quad n = 1, 2, \dots$$

Thus, the period of the Markov chain is 1, i.e., the Markov chain is aperiodic.

Problem 3

Consider a continuous-time Markov chain $\{X(t) : t \geq 0\}$ with state space $\mathcal{X} = \{1, 2, 3\}$. The transition probability matrix of the built-in discrete time Markov chain is given by:

$$\boldsymbol{Q} = \begin{bmatrix} Q_{1,1} & Q_{1,2} & Q_{1,3} \\ Q_{2,1} & Q_{2,2} & Q_{2,3} \\ Q_{3,1} & Q_{3,2} & Q_{3,3} \end{bmatrix} = \begin{bmatrix} 0 & p & q \\ q & 0 & p \\ p & q & 0 \end{bmatrix}$$

(Continued on page 6.)

where $0 < p < 1$, $0 < q < 1$ and $p + q = 1$.

The amount of time spent in state i is exponentially distributed with rate λ_i, $i = 1, 2, 3$, and we let:

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

For all $i, j \in \mathcal{X}$ we let:

$q_{ij} = \lambda_i Q_{ij} =$ The transition rate from state i to state j if $i \neq j$.

Finally, we let the matrix $\boldsymbol{R}$ be given by:

$$\boldsymbol{R} = \begin{bmatrix} -\lambda_1 & q_{1,2} & q_{1,3} \\ q_{2,1} & -\lambda_2 & q_{2,3} \\ q_{3,1} & q_{3,2} & -\lambda_3 \end{bmatrix}$$

a) Show that:

$$\boldsymbol{R} = \boldsymbol{\Lambda}(\boldsymbol{Q} - \boldsymbol{I}),$$

where:

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

SOLUTION:

We start out by noting that by definition we have:

$$q_{ij} = \lambda_i Q_{ij}, \qquad \text{for all } i, j \in \mathcal{X}.$$

Hence, it follows that:

$$\boldsymbol{R} = \begin{bmatrix} -\lambda_1 & q_{1,2} & q_{1,3} \\ q_{2,1} & -\lambda_2 & q_{2,3} \\ q_{3,1} & q_{3,2} & -\lambda_3 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & \lambda_1 Q_{1,2} & \lambda_1 Q_{1,3} \\ \lambda_2 Q_{2,1} & -\lambda_2 & \lambda_2 Q_{2,3} \\ \lambda_3 Q_{3,1} & \lambda_3 Q_{3,2} & -\lambda_3 \end{bmatrix}$$

On the other hand, since $Q_{ii} = 0$, for $i = 1, 2, 3$, we have:

$$\boldsymbol{\Lambda}(\boldsymbol{Q} - \boldsymbol{I}) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \cdot \begin{bmatrix} -1 & Q_{1,2} & Q_{1,3} \\ Q_{2,1} & -1 & Q_{2,3} \\ Q_{3,1} & Q_{3,2} & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda_1 & \lambda_1 Q_{1,2} & \lambda_1 Q_{1,3} \\ \lambda_2 Q_{2,1} & -\lambda_2 & \lambda_2 Q_{2,3} \\ \lambda_3 Q_{3,1} & \lambda_3 Q_{3,2} & -\lambda_3 \end{bmatrix}$$

Thus, we conclude that:

$$\boldsymbol{R} = \boldsymbol{\Lambda}(\boldsymbol{Q} - \boldsymbol{I}),$$

(Continued on page 7.)

b) Assume that $\boldsymbol{\rho} = (\rho_1, \rho_2, \rho_3)$ is a vector such that:

$$\boldsymbol{\rho}\boldsymbol{Q} = \boldsymbol{\rho}$$

and let $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \kappa_3) = \boldsymbol{\rho}\boldsymbol{\Lambda}^{-1}$. Show that:

$$\boldsymbol{\kappa}\boldsymbol{R} = \boldsymbol{0}$$

SOLUTION:

By using the result from (a) and that $\boldsymbol{\kappa} = \boldsymbol{\rho}\boldsymbol{\Lambda}^{-1}$ we get:

$$\begin{aligned}\boldsymbol{\kappa}\boldsymbol{R} &= \boldsymbol{\rho}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}(\boldsymbol{Q} - \boldsymbol{I}) = \boldsymbol{\rho}(\boldsymbol{Q} - \boldsymbol{I}) \\ &= \boldsymbol{\rho}\boldsymbol{Q} - \boldsymbol{\rho} = \boldsymbol{0}.\end{aligned}$$

c) We now introduce:

$$P_{ij}(t) = P(X(t) = j | X(0) = i), \quad \text{for all } i, j \in \mathcal{X},$$

and let $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$, where:

$$\pi_j = \lim_{t \to \infty} P_{ij}(t), \quad \text{for all } j \in \mathcal{X},$$

assuming that the limits exist.

Kolmogorov's forward equations can be written as:

$$\boldsymbol{P}'(t) = \boldsymbol{P}(t)\boldsymbol{R},$$

where:

$$\boldsymbol{P}(t) = \begin{bmatrix} P_{1,1}(t) & P_{1,2}(t) & P_{1,3}(t) \\ P_{2,1}(t) & P_{2,2}(t) & P_{2,3}(t) \\ P_{3,1}(t) & P_{3,2}(t) & P_{3,3}(t) \end{bmatrix}$$

Use this to show that $\boldsymbol{\pi}$ must satisfy the following set of equations:

$$\boldsymbol{\pi}\boldsymbol{R} = \boldsymbol{0}$$

SOLUTION:

Since we have assumed that the above limits exist, it follows that:

$$\lim_{t \to \infty} P'_{ij}(t) = 0, \quad \text{for all } i, j \in \mathcal{X}.$$

Hence, by taking the limit, the left-hand side of Kolmogorov's forward equations becomes:

$$\lim_{t \to \infty} \boldsymbol{P}'(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(Continued on page 8.)

Furthermore, the right-hand side of Kolmogorov's forward equations becomes:

$$\lim_{t\to\infty} \boldsymbol{P}(t)\boldsymbol{R} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \cdot \boldsymbol{R}$$

Hence, we get that:

$$\boldsymbol{\pi}\boldsymbol{R} = \boldsymbol{0}$$

d) Show that:

$$\pi_j = \frac{\lambda_j^{-1}}{\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1}}, \quad j = 1, 2, 3.$$

[Hint: Substitute $y_j = \lambda_j \pi_j$, $j = 1, 2, 3$ in the equations.]

SOLUTION:

The limit distribution $\boldsymbol{\pi}$ can be found by solving the set of linear equations $\boldsymbol{\pi}\boldsymbol{R} = \boldsymbol{0}$ combined with the equation $\pi_1 + \pi_2 + \pi_3 = 1$.

These equations can be written as:

$$-\lambda_1\pi_1 + \lambda_2 q\pi_2 + \lambda_3 p\pi_3 = 0$$

$$\lambda_1 p\pi_1 - \lambda_2\pi_2 + \lambda_3 q\pi_3 = 0$$

$$\lambda_1 q\pi_1 + \lambda_2 p\pi_2 - \lambda_3\pi_3 = 0$$

We simplify the equations by substituting $y_j = \lambda_j \pi_j$, $j = 1, 2, 3$:

$$-y_1 + qy_2 + py_3 = 0$$

$$py_1 - y_2 + qy_3 = 0$$

$$qy_1 + py_2 - y_3 = 0$$

By 1(d) these equations are satisfied if $y_1 = y_2 = y_3$. Hence, by substituting back we get that:

$$\lambda_1\pi_1 = \lambda_2\pi_2 = \lambda_3\pi_3$$

This implies that $\boldsymbol{\pi} = c(\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1})$ where c is a constant. Finally, we determine c so that $\pi_1 + \pi_2 + \pi_3 = 1$, and get:

$$\pi_j = \frac{\lambda_j^{-1}}{\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1}}, \quad j = 1, 2, 3.$$

(Continued on page 9.)

Alternatively, we observe that the matrix $\boldsymbol{Q}$ is *doubly stochastic*. Thus, it follows that if $\boldsymbol{\rho} = c(1,1,1)$, where c is a constant, then:

$$\boldsymbol{\rho}\boldsymbol{Q} = \boldsymbol{\rho}$$

We proceed by letting:

$$\boldsymbol{\kappa} = \boldsymbol{\rho}\boldsymbol{\Lambda}^{-1} = c(1,1,1) \cdot \begin{bmatrix} \lambda_1^{-1} & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_3^{-1} \end{bmatrix} = c(\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1})$$

Then by the result in (b) it follows that:

$$\boldsymbol{\kappa}\boldsymbol{R} = \boldsymbol{0}$$

Finally, we determine c so that $\kappa_1 + \kappa_2 + \kappa_3 = 1$. That is, c must satisfy:

$$c(\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1}) = 1.$$

Thus, we get that:

$$c = (\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1})^{-1}$$

With this c-value the vector $\boldsymbol{\kappa}$ satisfies all conditions for the limit distribution. That is, $\boldsymbol{\pi} = \boldsymbol{\kappa}$, or more specifically:

$$\pi_j = \frac{\lambda_j^{-1}}{\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1}}, \quad j = 1,2,3,$$

as before.

Problem 4

Let $\{N(t) : t \geq 0\}$ be a *renewal process* with *interarrival times* $X_1, X_2, \ldots$. The *renewal times*, denoted by $S_0, S_1, \ldots$, are given by:

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i, \quad n = 1,2,\ldots$$

The cumulative distribution function of the interarrival times is denoted by F, and we let $\bar{F}(t) = 1 - F(t)$.

a) Show that:

$$P(N(t) = n) = \int_0^t \bar{F}(t-s) f_{S_n}(s) ds, \quad n = 1,2,\ldots$$

where f_{S_n} denotes the density function of S_n, $n = 1,2,\ldots$

(Continued on page 10.)

SOLUTION:

$P(N(t) = n)$ can be calculated by conditioning on S_n:

$$\begin{aligned} P(N(t) = n) &= \int_0^{\infty} P(N(t) = n | S_n = s) f_{S_n}(s) ds \\ &= \int_0^{t} P(N(t) = n | S_n = s) f_{S_n}(s) ds + \int_t^{\infty} 0 \cdot f_{S_n}(s) ds \\ &= \int_0^{t} P(X_{n+1} > t - s | S_n = s) f_{S_n}(s) ds \\ &= \int_0^{t} \bar{F}(t - s) f_{S_n}(s) ds \end{aligned}$$

b) Assume that $X_1, X_2, \ldots$ are independent and exponentially distributed with rate λ. Explain briefly why this implies that:

$$f_{S_n}(s) = \frac{\lambda^n}{(n-1)!} s^{n-1} e^{-\lambda s}, \quad s > 0, \quad n = 1, 2, \ldots$$

and use this to show that:

$$P(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 0, 1, \ldots$$

SOLUTION:

If $X_1, X_2, \ldots$ are independent and exponentially distributed with rate λ it follows that:

$$S_n \sim Gamma(n, \lambda), \quad n = 1, 2, \ldots.$$

Thus, the density of S_n is:

$$f_{S_n}(s) = \frac{\lambda^n}{(n-1)!} s^{n-1} e^{-\lambda s}, \quad s > 0, \quad n = 1, 2, \ldots$$

We proceed by using the result from (a), and get:

$$\begin{aligned} P(N(t) = n) &= \int_0^{t} e^{-\lambda(t-s)} \cdot \frac{\lambda^n}{(n-1)!} s^{n-1} e^{-\lambda s} ds \\ &= \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \int_0^t s^{n-1} ds \\ &= \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \left[\frac{1}{n} s^n \right]_0^t \\ &= \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 1, 2, \ldots \end{aligned}$$

(Continued on page 11.)

Finally, we have:

$$P(N(t) = 0) = P(X_1 > t) = e^{-\lambda t} = \frac{(\lambda t)^0}{0!} e^{-\lambda t}, \quad t > 0.$$

END