UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	STK2130 — Modelling by stochastic processes
Day of examination:	Monday 13. June 2022
Examination hours:	09:00 – 13:00

This problem set consists of 3 pages.

Appendices:	Formulae note for STK1100 and STK1110
Permitted aids:	Formulae note for STK1100 and STK1110 Accepted calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 *Discrete-time Markov chains* (weight 35%)

Consider a discrete-time Markov chain with state space $\mathcal{S} = \{0, 1, 2, 3\}$ and the one-step transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 2q & p & 0 & 0 \\ 3p & q & 0 & 0 \\ 0 & 1/5 & 2/5 & 2/5 \\ 3/5 & 0 & 0 & 2/5 \end{pmatrix}.$$

1a (weight 5%)

Determine p and q.

1b (weight 5%)

Draw the state diagram of the Markov chain.

1c (weight 5%)

Compute the two-step transition probability matrix and find

$$\mathbb{P}\left(X_3 = 0,\ X_2 = 3 \mid X_0 = 2\right).$$

1d (weight 5%)

Find

$$\mathbb{P}(X_k \in \{0, 1\} \text{ for some } 1 \le k \le 4 \mid X_0 = 2),$$

i.e. the probability that $\{X_n\}$ will visit $\{0, 1\}$ at least once within 4 steps given that it starts at state 2.

1e (weight 5%)

Determine the classes of the Markov chain. Give the definition of recurrent and transient states and, for each class, determine whether it is transient or recurrent.

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1f (weight 5%)

Conditioned upon the chain has entered one of the states 0 or 1, find the stationary distribution over these two states.

1g (weight 5%)

Let $s_{2,3}$ denote the expected number of visits to state 3 given that $X_0 = 2$. Find $s_{2,3}$.

Problem 2 *Poisson process* (weight 35%)

2a (weight 5%)

Give **both** definitions of the homogeneous Poisson process. Provide the explicit formulas for the distribution of the increments.

The police station is open 24 hours a day, 7 days a week, and wants to determine how many employees should be responding to phone calls. In order to do that, it is necessary to analyze the number of incoming phone calls every day.

Assume that phone calls arrive according to a homogeneous Poisson process with the rate $\lambda = 3$ calls per hour.

2b (weight 5%)

Find the expected number of calls

- from 18:00 to 6:00,
- in total during the day, i.e. from 00:00 to 24:00.

2c (weight 5%)

Find

$$\mathbb{P}\,(\text{there are no calls from 21:00 to 23:00})\,,$$
$$\mathbb{P}\,(\text{there are 2 or more calls from 10:00 to 15:00})\,.$$

2d (weight 5%)

Let S_{10} be the time of arrival of the 10th call. Write the density of S_{10}. How is this distribution called? Find $\mathbb{E}[S_{10}]$.

2e (weight 5%)

Some of the phone calls to the police do not require immediate actions (e.g. someone may be asking for an appointment) and some are urgent. The probability that a given call is urgent is $\frac{1}{3}$. What is the probability that there are no urgent calls during the night-time (from 18:00 to 6:00)? Find the expected number of urgent calls within this time period.

The homogeneous Poisson model described above implies that the number of received calls does not depend on the time of the day. It is not perfectly realistic since e.g. one would expect less phone calls

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at night. Therefore the police decided to consider a more realistic model and use a non-homogeneous Poisson process with rate ($t = 0$ corresponds to 00:00)

$$\lambda(t) = \begin{cases} 1, & t \in [0, 6), \\ -\frac{t^2}{6} + 4t - 17, & t \in [6, 18), \\ 1, & t \in [18, 24), \end{cases}$$

and which is periodic with period 24, i.e. $\lambda(t + 24) = \lambda(t)$.

2f (weight 5%)

Write the definition of the non-homogeneous Poisson process (either of two) and compare the expected number of calls over night from 18:00 to 6:00 with the expected number of calls from 6:00 to 18:00. What is the expected total number of calls during the entire day (i.e. from 00:00 to 24:00)?

2g (weight 5%)

What is the probability that the police station will get exactly 5 calls between 5:00 and 12:00?

Problem 3 *Continuous-time Markov chains* (weight 30%)

Let $\{N(t),\ t \geq 0\}$ be a standard Poisson process with parameter $\lambda > 0$.

3a (weight 10%)

Prove that $\{N(t),\ t \geq 0\}$ is a homogeneous Markov chain. Provide the explicit form of

$$P_{i,j}(t) := \mathbb{P}\left(N(s + t) = j \mid N(s) = i\right), \quad s,\ t \geq 0,$$

using the properties of the increments.

3b (weight 10%)

Determine the transition rate matrix $\mathbf{R}$ of the Poisson process $\{N(t),\ t \geq 0\}$.

3c (weight 10%)

Write the corresponding Kolmogorov forward and backward equations.