STK2130 Exam 2022 – Solutions

Problem 1 (35%)

Consider a discrete-time Markov chain with state space $\mathcal{S} = \{0, 1, 2, 3\}$ and the one-step transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 2q & p & 0 & 0 \\ 3p & q & 0 & 0 \\ 0 & 1/5 & 2/5 & 2/5 \\ 3/5 & 0 & 0 & 2/5 \end{pmatrix}.$$

1a (5%) *Determine p and q.*

Solution. The sum of elements in each row of $\mathbf{P}$ must be equal to 1, hence

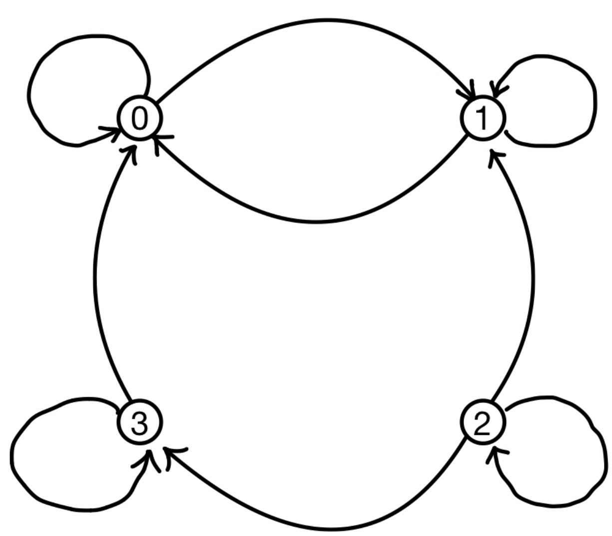
$$\begin{cases} 2q + p = 1 \\ 3p + q = 1 \end{cases} \Rightarrow \begin{cases} p = \frac{1}{5} \\ q = \frac{2}{5} \end{cases}$$

and thus

$$\mathbf{P} = \begin{pmatrix} 4/5 & 1/5 & 0 & 0 \\ 3/5 & 2/5 & 0 & 0 \\ 0 & 1/5 & 2/5 & 2/5 \\ 3/5 & 0 & 0 & 2/5 \end{pmatrix}.$$

1b (5%) *Draw the state diagram of the Markov chain.*

Solution.



1c (5%) *Compute the two-step transition probability matrix and find*

$$\mathbb{P}\left(X_3 = 0,\ X_2 = 3 \mid X_0 = 2\right).$$

Solution.

$$\mathbf{P}^{(2)} = \mathbf{P}^2 = \begin{pmatrix} 4/5 & 1/5 & 0 & 0 \\ 3/5 & 2/5 & 0 & 0 \\ 0 & 1/5 & 2/5 & 2/5 \\ 3/5 & 0 & 0 & 2/5 \end{pmatrix} \begin{pmatrix} 4/5 & 1/5 & 0 & 0 \\ 3/5 & 2/5 & 0 & 0 \\ 0 & 1/5 & 2/5 & 2/5 \\ 3/5 & 0 & 0 & 2/5 \end{pmatrix}$$
$$= \begin{pmatrix} 19/25 & 6/25 & 0 & 0 \\ 18/25 & 7/25 & 0 & 0 \\ 9/25 & 4/25 & 4/25 & 8/25 \\ 18/25 & 3/25 & 0 & 4/25 \end{pmatrix}$$
$$= \begin{pmatrix} 0.76 & 0.24 & 0 & 0 \\ 0.72 & 0.28 & 0 & 0 \\ 0.36 & 0.16 & 0.16 & 0.32 \\ 0.72 & 0.12 & 0 & 0.16 \end{pmatrix}$$

and

$$\begin{aligned}\mathbb{P}\left(X_3 = 0,\ X_2 = 3 \mid X_0 = 2\right) &= \frac{\mathbb{P}\left(X_3 = 0,\ X_2 = 3,\ X_0 = 2\right)}{\mathbb{P}(X_0 = 2)} \\ &= \frac{\mathbb{P}\left(X_3 = 0,\ X_2 = 3,\ X_0 = 2\right)}{\mathbb{P}(X_2 = 3, X_0 = 2)} \frac{\mathbb{P}(X_2 = 3, X_0 = 2)}{\mathbb{P}(X_0 = 2)} \\ &= \mathbb{P}\left(X_3 = 0 \mid X_2 = 3,\ X_0 = 2\right) \mathbb{P}(X_2 = 3 \mid X_0 = 2) \\ &= \mathbb{P}\left(X_3 = 0 \mid X_2 = 3\right) \mathbb{P}(X_2 = 3 \mid X_0 = 2) \\ &= P_{3,0} P^2_{2,3} = \frac{3}{5} * \frac{8}{25} = \frac{24}{125} = 0.192.\end{aligned}$$

1d (5%) *Find*

$$\mathbb{P}(X_k \in \{0,1\} \text{ for some } 1 \le k \le 4 \mid X_0 = 2),$$

i.e. the probability that $\{X_n\}$ *will visit* $\{0,1\}$ *at least once within* 4 *steps given that it starts at state* 2.

Solution. Consider a Markov chain $\{Y_n\}$ with state space $\{1,2,3\}$ and transition probability matrix

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 1/5 & 2/5 & 2/5 \\ 3/5 & 0 & 2/5 \end{pmatrix},$$

i.e. we "merged" two states 0 and 1 into a single absorbing state. Note that $\{X_n\}$ visits states $\{0,1\}$ within 4 steps if and only if $Y_4 = 1$. Whence wee need to find

$$Q^4_{2,1} := \mathbb{P}(Y_4 = 1 \mid Y_0 = 2).$$

That is easy:

$$\mathbf{Q}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 1/5 & 2/5 & 2/5 \\ 3/5 & 0 & 2/5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1/5 & 2/5 & 2/5 \\ 3/5 & 0 & 2/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 13/25 & 4/25 & 8/25 \\ 21/25 & 0 & 4/25 \end{pmatrix},$$
$$\mathbf{Q}^4 = \begin{pmatrix} 1 & 0 & 0 \\ 13/25 & 4/25 & 8/25 \\ 21/25 & 0 & 4/25 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 13/25 & 4/25 & 8/25 \\ 21/25 & 0 & 4/25 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0.872 & 0.0256 & 0.1024 \\ 0.9744 & 0 & 0.0256 \end{pmatrix},$$

whence

$$\mathbb{P}(X_k \in \{0,1\} \text{ for some } 1 \le k \le 4 \mid X_0 = 2) = Q^4_{2,1} = 0.872.$$

1e (5%) *Determine the classes of the Markov chain. Give the definition of recurrent and transient states and, for each class, determine whether it is transient or recurrent.*

Solution. From the diagram from **1b** it is clear that the Markov chain has three classes of communicating states:

- $\mathcal{C}_0 = \{0, 1\}$,
- $\mathcal{C}_1 = \{2\}$,
- $\mathcal{C}_2 = \{3\}$.

Definition. State i from the state space of the discrete-time Markov chain $\{X_n\}$ is called

- recurrent, if
$$\mathbb{P}\left(\bigcup_{r=1}^{\infty}\{X_r = i\} \,\middle|\, X_0 = i\right) = 1,$$
- transient, if
$$\mathbb{P}\left(\bigcup_{r=1}^{\infty}\{X_r = i\} \,\middle|\, X_0 = i\right) < 1.$$

Note also that recurrence/transience is a communicating class property; each finite and closed class (i.e. Markov chain stays in this class with probability 1 whenever it reaches it) is recurrent and if a class is not closed it must be transient.

Regarding the properties of our Markov chain, we have that:

- $\mathcal{C}_0 = \{0, 1\}$ is recurrent ($\{X_n\}$ stays in this class forever whenever reaches it),
- $\mathcal{C}_1 = \{2\}$ is transient ($\{X_n\}$ can leave this class with positive probability but never returns back with probability 1),
- $\mathcal{C}_2 = \{3\}$ is transient ($\{X_n\}$ can leave this class with positive probability but never returns back with probability 1).

1f (5%) *Conditioned upon the chain has entered one of the states* 0 *or* 1*, find the stationary distribution over these two states.*

Solution. When the process $\{X_n\}$ gets to 0 or 1, it stays in $\mathcal{C}_0$. Whence, after that moment, we can regard our Markov chain as the one with state space $\{0, 1\}$ and transition probability matrix

$$\widetilde{\mathbf{P}} = \begin{pmatrix} 4/5 & 1/5 \\ 3/5 & 2/5 \end{pmatrix}.$$

Moreover, this 2-state Markov chain is irreducible, aperiodic and positive recurrent (due to finite state space), therefore there exists a unique stationary distribution.

Denote the corresponding stationary probabilities π_0, π_1. They satisfy the system of equations

$$\begin{cases} \pi_0 = \frac{4}{5}\pi_0 + \frac{3}{5}\pi_1 \\ \pi_1 = \frac{1}{5}\pi_0 + \frac{2}{5}\pi_1 \\ \pi_0 + \pi_1 = 1 \end{cases} \Rightarrow \begin{cases} \pi_0 = \frac{3}{4} \\ \pi_1 = \frac{1}{4} \end{cases}.$$

1g (5%) *Let $s_{2,3}$ denote the expected number of visits to state* 3 *given that $X_0 = 2$. Find $s_{2,3}$.*

Solution. Both states 2 and 3 are transient, whence $s_{i,j}$, $i, j \in \{2, 3\}$ can be found as follows:

$$\begin{aligned} \begin{pmatrix} s_{2,2} & s_{2,3} \\ s_{3,2} & s_{3,3} \end{pmatrix} &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2/5 & 2/5 \\ 0 & 2/5 \end{pmatrix}\right)^{-1} \\ &= \begin{pmatrix} 3/5 & -2/5 \\ 0 & 3/5 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 5/3 & 10/9 \\ 0 & 5/3 \end{pmatrix}, \end{aligned}$$

i.e. $s_{2,3} = \frac{10}{9}$.

Problem 2 (35%)

2a (5%) *Give **both** definitions of the homogeneous Poisson process. Provide the explicit formulas for the distribution of the increments.*

Solution. Both definitions are given below.

Definition 1. A counting process $\{N(t),\ t \geq 0\}$ is called a homogeneous Poisson process with parameter $\lambda > 0$, if:

(i) $N(0) = 0$,

(ii) $\{N(t),\ t \geq 0\}$ has independent increments,

(iii) for any $t \geq 0$

$$\mathbb{P}(N(t+h) - N(t) = 1) = \lambda h + o(h), \quad h \to 0,$$

(iv) for any $t \geq 0$

$$\mathbb{P}(N(t+h) - N(t) \geq 2) = o(h), \quad h \to 0.$$

Definition 2. A counting process $\{N(t),\ t \geq 0\}$ is called a homogeneous Poisson process with parameter $\lambda > 0$, if:

(i) $N(0) = 0$,

(ii) $\{N(t),\ t \geq 0\}$ has independent increments,

(iii) for any $s, t \geq 0$ and $n \in \mathbb{N} \cup \{0\}$:

$$\mathbb{P}(N(s+t) - N(s) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t},$$

i.e. $N(s+t) - N(s)$ has Poisson distribution with parameter λt.

The police station is open 24 hours a day, 7 days a week, and wants to determine how many employees should be responding to phone calls. In order to do that, it is necessary to analyze the number of incoming phone calls every day.

Assume that phone calls arrive according to a homogeneous Poisson process with the rate $\lambda = 3$ calls per hour.

2b (5%) *Find the expected number of calls*

– *from 18:00 to 6:00,*

– *in total during the day, i.e. from 00:00 to 24:00.*

Solution. Since time period between 18:00 and 6:00 is 12 hours, the number of calls during that period has Poisson distribution with parameter $3 * 12 = 36$. Whence The expected number of calls during that period is 36. Similarly, the expected number of calls in total during the day is $3 * 24 = 72$.

2c (5%) *Find*

$$\mathbb{P}\left(\text{there are no calls from 21:00 to 23:00}\right),$$
$$\mathbb{P}\left(\text{there are 2 or more calls from 10:00 to 15:00}\right).$$

Solution. Since the increments have Poisson distribution, we have:

$$\begin{aligned}
\mathbb{P}\left(\text{there are no calls from 21:00 to 23:00}\right) &= \mathbb{P}(N(23) - N(21) = 0) = e^{-3*2} = e^{-6}, \\
\mathbb{P}\left(\text{there are 2 or more calls from 10:00 to 15:00}\right) &= \mathbb{P}(N(15) - N(10) \geq 2) \\
&= 1 - \mathbb{P}(N(15) - N(10) = 0) - \mathbb{P}(N(15) - N(10) = 1) \\
&= 1 - e^{-3*5} - 3 * 5e^{-3*5} = 1 - 16e^{-15}.
\end{aligned}$$

2d *Let S_{10} be the time of arrival of the 10th call. Write the density of S_{10}. How is this distribution called? Find $\mathbb{E}[S_{10}]$.*

Solution. The times between jumps of $\{N(t),\ t \geq 0\}$ are independent identically distributed random variables with exponential distribution with parameter $\lambda = 3$. Therefore, S_{10} is the sum of 10 independent exponentially distributed random variables with parameter $\lambda = 3$. Therefore, the distribution of S_{10} is Gamma distribution with the density

$$f(s) = \frac{\lambda^{10}}{\Gamma(10)} s^{10-1} e^{-\lambda s} \mathbb{1}_{s>0} = \frac{3^{10}}{9!} s^9 e^{-3s} \mathbb{1}_{s>0}.$$

In particular, $\mathbb{E}[S_{10}] = \frac{10}{3}$.

2e *Some of the phone calls to the police do not require immediate actions (e.g. someone may be asking for an appointment) and some are urgent. The probability that a given call is urgent is $\frac{1}{3}$. What is the probability that there are no urgent calls during the night-time (from 18:00 to 6:00)? Find the expected number of urgent calls within this time period.*

Solution. Urgent calls arrive according to a Poisson process with parameter $\frac{1}{3} * \lambda = \frac{1}{3} * 3 = 1$. Whence the number of urgent calls within the 12h time period between 18:00 and 6:00 has Poisson distribution with parameter 12. Whence

$$\mathbb{P}(\text{there are no urgent calls from 18:00 to 6:00}) = e^{-12}$$

and the expected number of urgent calls within this time period is 12.

The homogeneous Poisson model described above implies that the number of received calls does not depend on the time of the day. It is not perfectly realistic since e.g. one would expect less phone calls at night. Therefore the police decided to consider a more realistic model and use a non-homogeneous Poisson process with rate ($t = 0$ corresponds to 00:00)

$$\lambda(t) = \begin{cases} 1, & t \in [0, 6), \\ -\frac{t^2}{6} + 4t - 17, & t \in [6, 18), \\ 1, & t \in [18, 24). \end{cases}$$

2f *Write the definition of the non-homogeneous Poisson process (either of two) and compare the expected number of calls over night from 18:00 to 6:00 with the expected number of calls from 6:00 to 18:00. What is the expected total number of calls during the day (i.e. from 00:00 to 24:00)?*

Solution. Either of the following two definitions is fine.

Definition 1. A counting process $\{N(t),\ t \geq 0\}$ is called a homogeneous Poisson process with (functional) parameter $\lambda = \lambda(t)$, if:

(i) $N(0) = 0$,

(ii) $\{N(t),\ t \geq 0\}$ has independent increments,

(iii) for any $t \geq 0$

$$\mathbb{P}(N(t+h) - N(t) = 1) = \lambda(t)h + o(h), \quad h \to 0,$$

(iv) for any $t \geq 0$

$$\mathbb{P}(N(t+h) - N(t) \geq 2) = o(h), \quad h \to 0.$$

Definition 2. A counting process $\{N(t),\ t \geq 0\}$ is called a homogeneous Poisson process with (functional) parameter $\lambda = \lambda(t)$, if:

(i) $N(0) = 0$,

(ii) $\{N(t),\ t \geq 0\}$ has independent increments,

(iii) for any $s, t \geq 0$ and $n \in \mathbb{N} \cup \{0\}$:

$$\mathbb{P}(N(s+t) - N(s) = n) = \frac{\left(\int_s^{s+t} \lambda(u)du\right)^n}{n!} e^{-\int_s^{s+t} \lambda(u)du},$$

i.e. $N(s+t) - N(s)$ has Poisson distribution with parameter $\int_s^{s+t} \lambda(u)du$.

The number of calls over night from 18:00 to 6:00 has thus Poisson distribution with parameter $\int_{18}^{24} \lambda(u)du + \int_0^6 \lambda(u)du = 12$ whereas the number of calls during the day from 6:00 to 18:00 is

$$\int_6^{18} \lambda(u)du = \int_6^{18} \left(-\frac{u^2}{6} + 4u - 17\right) du = 60.$$

Whence the expected number of calls over night from 18:00 to 6:00 is 12 and it is much less than the expected number of calls during the day from 6:00 to 18:00 that is equal to 60. The total expected number of calls (24h) is then

$$\int_0^{24} \lambda(u)du = 12 + 60 = 72.$$

2g What is the probability that the police station will get exactly 5 calls between 5:00 and 12:00?

Solution. The number of calls between 5:00 and 12:00 has Poisson distribution with parameter

$$\int_5^{12} \lambda(u)du = \int_5^6 1du + \int_6^{12} \left(-\frac{u^2}{6} + 4u - 17\right) du = 1 + 30 = 31.$$

Whence

$$\mathbb{P}(\text{there are exactly 5 calls between 5:00 and 12:00}) = \frac{31^5}{5!} e^{-31}.$$

Problem 3 (30%)

Let $\{N(t),\ t \geq 0\}$ be a standard Poisson process with parameter $\lambda > 0$.

3a (10%) *Prove that $\{N(t),\ t \geq 0\}$ is a homogeneous Markov chain. Provide the explicit form of*

$$P_{i,j}(t) := \mathbb{P}\left(N(s+t) = j \mid N(s) = i\right), \quad s,\ t \geq 0,$$

using the properties of the increments.

Solution. Take arbitrary $0 \leq t_1 < t_2 < ... < t_k < s < t + s$ and states $n_1 \leq n_2 \leq ... \leq n_k \leq i \leq j$ (since the paths of $\{N(t),\ t \geq 0\}$ are non-decreasing, it is sufficient to check Markov property only for a monotone sequence of states) and consider

$$\begin{aligned}
&\mathbb{P}(N(s+t) = j \mid N(t_1) = n_1,\ ...,\ N(t_k) = n_k,\ N(s) = i) \\
&= \frac{\mathbb{P}(N(s+t) = j,\ N(t_1) = n_1,\ ...,\ N(t_k) = n_k,\ N(s) = i)}{\mathbb{P}(N(t_1) = n_1,\ ...,\ N(t_k) = n_k,\ N(s) = i)} \\
&= \frac{\mathbb{P}(N(t_1) = n_1,\ N(t_2) - N(t_1) = n_2 - n_1,\ ...,\ N(s) - N(t_k) = i - n_k,\ N(s+t) - N(s) = j - i)}{\mathbb{P}(N(t_1) = n_1,\ N(t_2) - N(t_1) = n_2 - n_1,\ ...,\ N(s) - N(t_k) = i - n_k)}.
\end{aligned}$$

Since the increments of $\{N(t),\ t \geq 0\}$ are independent, we can write

$$\begin{aligned}
&\frac{\mathbb{P}(N(t_1) = n_1,\ N(t_2) - N(t_1) = n_2 - n_1,\ ...,\ N(s) - N(t_k) = i - n_k,\ N(s+t) - N(s) = j - i)}{\mathbb{P}(N(t_1) = n_1,\ N(t_2) - N(t_1) = n_2 - n_1,\ ...,\ N(s) - N(t_k) = i - n_k)} \\
&= \frac{\mathbb{P}(N(t_1) = n_1)\mathbb{P}(N(t_2) - N(t_1) = n_2 - n_1) \cdots \mathbb{P}(N(s) - N(t_k) = i - n_k)\mathbb{P}(N(s+t) - N(s) = j - i)}{\mathbb{P}(N(t_1) = n_1)\mathbb{P}(N(t_2) - N(t_1) = n_2 - n_1) \cdots \mathbb{P}(N(s) - N(t_k) = i - n_k)} \\
&= \mathbb{P}(N(s+t) - N(s) = j - i),
\end{aligned}$$

i.e.

$$\mathbb{P}(N(s+t)=j \mid N(t_1)=n_1,\ ...,\ N(t_k)=n_k,\ N(s)=i)=\mathbb{P}(N(s+t)-N(s)=j-i).$$

Similarly,

$$\begin{aligned}\mathbb{P}(N(s+t)=j \mid N(s)=i) &= \frac{\mathbb{P}(N(s+t)=j,\ N(s)=i)}{\mathbb{P}(N(s)=i)} = \frac{\mathbb{P}(N(s+t)-N(s)=j-i,\ N(s)=i)}{\mathbb{P}(N(s)=i)} \\ &= \frac{\mathbb{P}(N(s+t)-N(s)=j-i)\mathbb{P}(N(s)=i)}{\mathbb{P}(N(s)=i)} \\ &= \mathbb{P}(N(s+t)-N(s)=j-i).\end{aligned}$$

Therefore,

$$\mathbb{P}(N(s+t)=j \mid N(t_1)=n_1,\ ...,\ N(t_k)=n_k,\ N(s)=i)=\mathbb{P}(N(s+t)=j \mid N(s)=i)$$

which proves that $\{N(t),\ t\geq 0\}$ is a Markov chain.

Next, by the property of increments of the Poisson process,

$$\mathbb{P}(N(s+t)=j \mid N(s)=i)=\begin{cases}\frac{(\lambda t)^{j-i}}{(j-i)!}e^{-\lambda t}, & \text{if } j\geq i,\\ 0, & \text{otherwise.}\end{cases}$$

We see that $\mathbb{P}(N(s+t)=j \mid N(s)=i)$ does not depend on s, whence the Markov chain $\{N(t),\ t\geq 0\}$ is homogeneous.

Finally, as mentioned above, for any $i,j\in\mathbb{N}\cup\{0\}$,

$$P_{i,j}(t)=\begin{cases}\frac{(\lambda t)^{j-i}}{(j-i)!}e^{-\lambda t}, & \text{if } j\geq i,\\ 0, & \text{otherwise.}\end{cases}$$

3b (10%) *Determine the transition rate matrix* $\mathbf{R}$ *of the Poisson process* $\{N(t),\ t\geq 0\}$.

Solution. For a Poisson process, the time between subsequent transitions is exponentially distributed with fixed parameter λ that is constant for all states. Moreover, the built-in discrete-time Markov chain has the following transition probabilities:

$$Q_{i,j}=\begin{cases}1, & j=i+1,\\ 0, & \text{otherwise.}\end{cases}$$

Therefore the transition rate matrix has the form

$$\mathbf{R}=\begin{pmatrix}-\lambda & \lambda & 0 & 0 & 0 & \cdots\\ 0 & -\lambda & \lambda & 0 & 0 & \cdots\\ 0 & 0 & -\lambda & \lambda & 0 & \cdots\\ 0 & 0 & 0 & -\lambda & \lambda & \cdots\\ 0 & 0 & 0 & 0 & -\lambda & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

3c (10%) *Write the corresponding Kolmogorov forward and backward equations.*

Solution. Given the matrix $\mathbf{R}$ from **3b**, it is easy to write down the corresponding Kolmogorov forward and backward equations:

- *backward*:

$$P'_{i,j}(t)=\lambda P_{i+1,j}(t)-\lambda P_{i,j}(t), \qquad i,j\in\mathbb{N}\cup\{0\},$$

- *forward*:

$$\begin{aligned}P'_{i,0}(t)&=-\lambda P_{i,0}(t), && i\in\mathbb{N}\cup\{0\},\\ P'_{i,j}(t)&=\lambda P_{i,j-1}(t)-\lambda P_{i,j}(t), && i\in\mathbb{N}\cup\{0\},\ j\in\mathbb{N}.\end{aligned}$$