UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK2130 — Modelling by Stochastic Processes.

Day of examination: Thursday, June 8th, 2023.

Examination hours: 09.00 – 13.00.

This problem set consists of 2 pages.

Appendices: List of formulas for STK1100 and STK1110.

Permitted aids: Accepted calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All items (Problems 1a, 1b etc.) count equally. If there is a problem you cannot solve, you may still use the result in the sequel. All answers have to be substantiated.

Problem 1 (50 points) A Markov chain X has state space $S = \{0, 1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

a) Draw a state diagram which shows how the states communicate.

b) Show that X is ergodic (i.e. irreducible, recurrent, and aperiodic).

c) Show that the limit probabilities of X are $\pi_0 = \pi_1 = \pi_3 = \frac{3}{13}$, $\pi_2 = \frac{4}{13}$.

d) Is the Markov chain X reversible?

e) Assume that X starts in state 0. How many times will it in average hit state 2 before it hits state 3? Depending on how you solve the problem, you may or may not need that

$$\begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{2} & 1 & 1 \\ 1 & 2 & \frac{4}{3} \\ \frac{1}{2} & 1 & \frac{5}{3} \end{pmatrix}$$

Problem 2 (60 points) In this problem we shall study how taxis and customers arrive at a taxi stop. We assume that at time 0 the stop is empty (i.e. no taxis and no passengers), that taxis arrive according to a Poisson process N_T with rate λ, and that customers arrive according to a Poisson process N_C with rate μ. We assume that the two Poisson processes are independent, and for simplicity we also assume that it takes no time to get into a taxi: Whenever a taxi and a customer meet at the stop, the passenger gets in and the taxi leaves instantaneously. There are never more than one

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passenger in each taxi (if you want to be more realistic, you can just count a company of several people as one customer).

We count time in minutes and assume that $\lambda = 0.2$ and $\mu = 0.3$.

a) How long does it take in average before the first taxi arrives?

b) What is the probability that the first customer arrives before the first taxi? What is the probability that the second customer arrives before the first taxi?

c) Call it an *event* when a taxi or a passenger arrives. What is the expected time for the first event to occur?

d) Let $N(t)$ be the number of events that have occurred before time t. What kind of process is N? What is the probability that $N(10) = 5$?

e) What is the probability that the K-th event is that a taxi arrives?

f) What is the probability that the 32nd event is that a taxi arrives, picks up a customer and leaves 2 other customers still waiting? (You don't need to find a decimal number; a mathematical expression suffices).

Problem 3 (20 points) Assume that B is a standard Brownian motion.

a) Show that if $u \geq v$, then $E[B(u)B(v)] = v$. Also show that for $u > v$, the linear combination $\alpha B(u) + \beta B(v)$ is normally distributed for all nonzero $\alpha, \beta \in \mathbb{R}$ (you may use freely that a linear combination of two *independent*, normally distributed random variables is normally distributed).

One can show that the process

$$X(t) = \begin{cases} tB(\frac{1}{t}) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

is also a standard Brownian motion. We shall prove part of this.

b) Assume that $t \geq s$. Show that $E[X(t)X(s)] = s$. Show also that $X(t) - X(s)$ is normally distributed with mean 0 and variance $t - s$ (this is actually enough to conclude that X is a Brownian motion if one knows a little more theory).

THE END