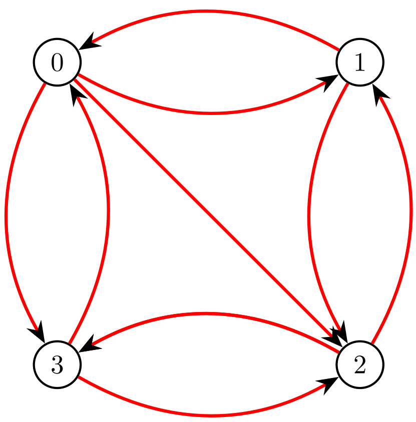
STK2130: Solution to the exam spring 2023

Problem 1 a) The state diagram:



b) From the diagram we see that it is possible to get from any state to any other state by following the arrows, so the Markov chain is irreducible. It is also recurrent as any irreducible Markov chain with a finite state space is recurrent. To see that it is aperiodic, note that it is possible to start in state 2 and be back in state 2 after 2 as well as after 3 steps. This means that a period would have to divide both 2 and 3, which is impossible. Hence state 2 is aperiodic, and since being aperiodic is a class property, all states are aperiodic, and hence X is aperiodic.

c) The limit probabilities $\vec{\pi} = (\pi_0, \pi_1, \pi_2, \pi_3)$ have to satisfy the equation $\vec{\pi}P = \vec{\pi}$. We check with $\vec{\pi} = (\frac{3}{13}, \frac{3}{13}, \frac{4}{13}, \frac{3}{13})$ and see that:

$$\vec{\pi}P = \left(\frac{3}{13}, \frac{3}{13}, \frac{4}{13}, \frac{3}{13}\right)\begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} = \left(\frac{3}{13}, \frac{3}{13}, \frac{4}{13}, \frac{3}{13}\right)$$

As the Markov chain is ergodic, and the π_i's are positive and satisfy $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$, we know from the theory that they are the limit probabilities.

d) If the Markov chain is reversible, it satisfies the detailed balance equation $\pi_i P_{ij} = \pi_j P_{ji}$ for all states i and j. As we have $\pi_1 P_{12} = \frac{3}{13} \cdot \frac{1}{2} = \frac{3}{26}$ and $\pi_2 P_{21} = \frac{4}{13} \cdot \frac{1}{2} = \frac{4}{26}$, we se that $\pi_1 P_{12} \neq \pi_2 P_{21}$, and hence the Markov chain is not reversible.

e) We give two solutions of this problem:

Solution 1: Let us modify the Markov chain such that state 3 becomes absorbing. The new transition matrix is

$$P' = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that the number of times the original process hits 2 before it hits 3, is the same as the number of times the modified process X' hits 2.

If we only consider transitions between the (now) transient states 0, 1, and 2, we get the reduced matrix

$$P'_T = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Hence

$$I - P'_T = \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

and (using the formula in the problem)

$$S = (I - P_T)^{-1} = \begin{pmatrix} \frac{3}{2} & 1 & 1 \\ 1 & 2 & \frac{4}{3} \\ \frac{1}{2} & 1 & \frac{5}{3} \end{pmatrix}$$

According to the theory, s_{ij} is the average number of times X' started in i hits j. As $s_{02} = 1$, X started at 0 will in average hit 2 once before it hits 3.

Solution 2: Let t_i, $i = 0, 1, 2$, be the number of times the process started in state i hits 2 before it hits 3. By looking at the transitions, we see that

$$t_0 = \frac{1}{3}t_1 + \frac{1}{3}t_2$$

$$t_1 = \frac{1}{2}t_0 + \frac{1}{2}t_2$$

$$t_2 = 1 + \frac{1}{2}t_1$$

(the 1 in the last equation is due to the fact that we are now considering what is happening when we start in state 2, and hence have to count the starting position as a visit). Rearranging these equations, we get

$$t_0 - \frac{1}{3}t_1 - \frac{1}{3}t_2 = 0$$

$$-\frac{1}{2}t_0 + t_1 - \frac{1}{2}t_2 = 0$$

$$-\frac{1}{2}t_1 + t_2 = 1,$$

which on matrix form becomes

$$\begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Using that

$$\begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{2} & 1 & 1 \\ 1 & 2 & \frac{4}{3} \\ \frac{1}{2} & 1 & \frac{5}{3} \end{pmatrix},$$

we get

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 1 & 1 \\ 1 & 2 & \frac{4}{3} \\ \frac{1}{2} & 1 & \frac{5}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{4}{3} \\ \frac{5}{3} \end{pmatrix}$$

and hence $t_0 = 1$ (there are, of course, many other ways to solve the system of equations).

Problem 2 a) The waiting time T_1^t for taxis is an exponentially distributed random variable with rate 0.2. The expectation is $E[T_1^t] = \frac{1}{\lambda} = \frac{1}{0.2} = 5$, which means that the expected waiting time is 5 minutes.

b) T_1^t and T_1^c are two independent exponential random variables with rates λ and μ, respectively. The probability that T_1^c is the smallest, is $\frac{\mu}{\lambda+\mu} = \frac{0.3}{0.2+0.3} = 0.6$. Hence the probability that a customer arrives first is 0.6. When the first customer has arrived, the processes start over again as exponential distributions are memoryless, and hence the probability that the next one to arrive is a customer, is still 0.6. This means that the probability that the first two to arrive are customers, is $0.6 \cdot 0.6 = 0.36$.

c) The minimum of two independent exponential random variables with rates λ and μ is a new exponential random variable with rate $\lambda + \mu = 0.2 + 0.3 = 0.5$. Hence the waiting time is $\frac{1}{0.5} = 2$ minutes.

d) As the random waiting times for new events are exponentially distributed with rate 0.5, N is a Poisson process with rate 0.5. This means that $N(10)$ is Poisson distributed with mean $0.5 \cdot 10 = 5$, and hence

$$P[N(10) = 5] = \frac{5^5}{5!}e^{-5} = \frac{625}{24}e^{-5} \approx 0.175$$

(you don't need the decimal number to get full score).

e) Let S be the (random) time of the $(K-1)$-st event. As the exponential distributions are memoryless, the waiting time for the first taxi at time S is exponential with rate λ, and the waiting time for the first customer at time S is exponential with rate μ, regardless of everything that has happened before. Hence the probability that a taxi is the first to arrive, is $\frac{\lambda}{\lambda+\mu} = \frac{0.2}{0.2+0.3} = 0.4$.

f) By e) the probability that the 32nd event is that a taxi arrives, is 0.4. If the taxi leaves 2 passengers behind, there must have been three customers waiting before it arrived. This means that there must have been 17 customer arrivals and 14 taxi arrivals among the 31 first events. As we can choose 14 among 31 in $\binom{31}{14}$ ways and each way has probability $0.4^{14}0.6^{17}$, the total probability is

$$0.4 \cdot \binom{31}{14} 0.4^{14} 0.6^{17} = \binom{31}{14} 0.4^{15} 0.6^{17}$$

Problem 3 a) Note that $B(u) = (B(u) - B(v)) + B(v)$ and that $B(u) - B(v)$ and $B(v)$ are independent with mean 0. Hence

$$E[B(u)B(v)] = E\left[\big((B(u) - B(v)) + B(v)\big)B(v)\right]$$

$$= E\left[(B(u) - B(v))B(v)\right] + E[B(v)^2] = E\left[(B(u) - B(v))\right] E\left[B(v)\right] + E[B(v)^2]$$

$$= 0 + v = v$$

By the same trick

$$\alpha B(u) + \beta B(v) = \alpha(B(u) - B(v)) + (\alpha + \beta)B(v),$$

which is a linear combination of two independent, normally distributed random variables, and hence normally distributed.

b) First note that if $s = 0$, we have $E[X(t)X(s)] = s$ as both expressions are zero. For $s > 0$, we have

$$E[X(t)X(s)] = E\left[tB\left(\frac{1}{t}\right)sB\left(\frac{1}{s}\right)\right] = stE\left[B\left(\frac{1}{t}\right)B\left(\frac{1}{s}\right)\right] = st\frac{1}{t} = s,$$

where we in the last step used a) (remember that if $t \geq s$, then $\frac{1}{t} \leq \frac{1}{s}$). Note that if we put $s = t$ in this expression, we get $E[X(t)^2] = t$ and $E[X(s)^2] = s$.

Next we observe that $X(t) - X(s) = tB(\frac{1}{t}) - sB(\frac{1}{s})$ is normally distributed by the second half of a), and that the mean is zero as both $B(\frac{1}{t})$ and $B(\frac{1}{s})$ have mean zero.

Finally, we have

$$\mathrm{Var}(X(t) - X(s)) = E\left[(X(t) - X(s))^2\right]$$
$$= E\left[X(t)^2\right] - 2E\left[X(t)X(s)\right] + E\left[X(s)^2\right] = t - 2s + s = t - s$$

where we have used that $E[X(t)X(s)] = s$, $E[X(t)^2] = t$ and $E[X(s)^2] = s$ as observed above.