Additional exercises for STK2130 Spring 2013.

Attention! For us and throughout the course, our transition probability matrices have been such that all rows sum up to 1, and the instantaneous rate matrices Q such that all rows sum up to 0, that is, we denote the matrices $P = (p_{ij})_{ij \in S}$ and $Q = (q_{ij})_{ij \in S}$. As you might have noticed, in exams 2004-07 the convention is the opposite, all matrices are transposed and they use the notation p_{ji} instead of ours p_{ij} (also matrix Q in exam 2004 corresponds to matrix R from Section 6.8 in the book).

Exercise 5

Here we will consider the special case of Problem 2 Exam 2004 or Example 6.13 from the book with "M" repair men but now $M = N = 2$.

The (infinitesimal) generator matrix in this special case is

$$R = \begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix}.$$

It can be shown that $R = H^{-1}\Lambda H$ where

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(\lambda+\mu) & 0 \\ 0 & 0 & -2(\lambda+\mu) \end{pmatrix}, \quad H = \begin{pmatrix} \mu^2 & 2\lambda\mu & \lambda^2 \\ -\mu & \mu-\lambda & \lambda \\ 1 & -2 & 1 \end{pmatrix},$$

$$H^{-1} = \frac{1}{(\lambda+\mu)^2} \begin{pmatrix} 1 & -2\lambda & \lambda^2 \\ 1 & \mu-\lambda & -\lambda\mu \\ 1 & 2\mu & \mu^2 \end{pmatrix}.$$

Let $P(t) = (p_{ij}(t))_{i,j \in \{0,1,2\}}$ be the matrix of transition probabilities. From the text book it is known that $P(t)$ can be expressed by R as

$$P(t) = e^{Rt} = I + \sum_{k=1}^{\infty} R^k \frac{t^k}{k!}.$$

a) Prove that

$$e^{\Lambda t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-(\lambda+\mu)t} & 0 \\ 0 & 0 & e^{-2(\lambda+\mu)t} \end{pmatrix}.$$

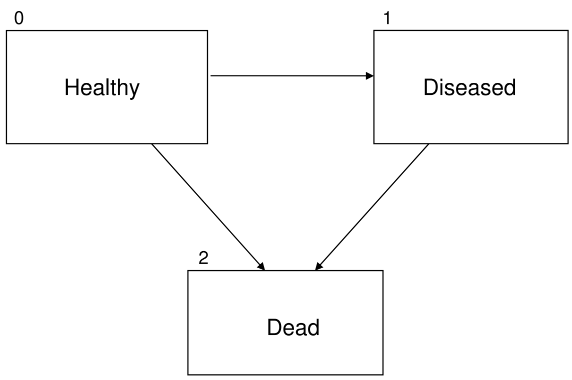
Hint: Use Taylor's expansion for the exponential function: $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$.

b) Express $P(t)$ as simple as possible by H and H^{-1}.

c) Find $\lim_{t\to\infty} P(t)$ expressed by λ and μ.

(d) Write down $p_{0j}(t)$, $j = 0, 1, 2$ in terms of λ and μ. Do you recognize the probability distribution?

Exercise 6 In a Healthy-Illness-Death (without recovery) scheme individuals may move from state 0. "Healthy" to state 1. "Illness" with rate q_{01}, from "Healthy" to state 2. "Death" with rate q_{02} and from Illness to Death with a rate q_{12}. No other direct transitions are possible. Let $X(t)$ be the state of such a process. The flow of the process is described in the figure below.



a) Write down the Kolmogorov equations for the transition probabilities $P_{ij}(t) = P(X(t) = j|X(0) = i)$ for such a system.

b) Find explicit expression for the $P_{ij}(t)$.

c) Plot $P_{00}(t), P_{01}(t)$ and $P_{02}(t)$ for $q_{01} = 1 = q_{02}$ and $q_{12} = 2$.

d) Set up the infinitesimal generator matrix $\mathbf{R} = \mathbf{P}'(0)$ where $\mathbf{P}'(t)$ is the matrix of the derivatives of the $P_{ij}(t)$.

e) Express the matrix $\mathbf{P(t)}$ consisting of the $P_{ij}(t)$ in terms of the eigenvalues $\mathbf{\Lambda}$ and matrix of eigen-vectors $\mathbf{U}$ of $\mathbf{R}$. Make a plot of $P_{00}(t), P_{01}(t)$ and $P_{02}(t)$ for $q_{01} = 1 = q_{02}$ and $q_{12} = 2$ using this representation.

f) We could extend the system by allowing for recovery by introducing a transition rate $q_{10} > 0$. Write down the matrix $\mathbf{R} = \mathbf{P}'(0)$ for this system and make a plot of $P_{00}(t), P_{01}(t)$ and $P_{02}(t)$ for $q_{01} = 1 = q_{02}, q_{12} = 2$ and $q_{10} = 1$ for this system using the corresponding method as in question e).