Exercise 4.20

A transition matrix P is said to be doubly stochastic if the sum over each column equals 1. If such a chain is irreducible and aperiodic and consists of $M+1$ states $0, 1, \ldots, M$ show that the limiting probabilities are given by $\pi_j = \frac{1}{M+1}$ for all $j = 0, \ldots, M$.

Solution: The assumption of irreducibility means that all states communicate with each other and aperiodic means that all states are aperiodic, i.e.: For a sufficient large n $p_{ii}^{(n)} > 0$ for all i.

To find the limiting probabilities we have to solve the following system

$$\pi = \pi P \iff \pi(P - id) = 0$$

or

$$(P^t - id)\pi = 0.$$

A doubly stochastic matrix has eigenvalue 1 indeed! Because if we compute $P - id$, then each column sums up to 0! So $\det(P - id) = 0$ this means that the rank is not $M+1$. Moreover, observe that the matrix can not have rank less than M.

Then, since the independent variables are all 0, (the overall matrix rank, $(P - id|0)$, does not change) the system is compatible, but may have infinitely many solutions: since $\text{rank}(P - id) = M$ and $\text{rank}(P - id|0) = M$ with $M+1$ unknowns.

Nevertheless, since we want those solutions such that $\sum_{j=0}^{M} \pi_j = 1$, this condition makes the system have a unique solution! (we add a row of 1's in the matrix of the system) that is, we increase the ranks by 1.

Claim: The vector $\pi = (\frac{1}{M+1}, \ldots, \frac{1}{M+1})$, satisfies the system: Indeed,

$$\begin{aligned}
\left(\frac{1}{M+1}, \ldots, \frac{1}{M+1}\right)\begin{pmatrix} p_{00} & \cdots & p_{0M} \\ \vdots & \ddots & \vdots \\ p_{M0} & \cdots & p_{MM} \end{pmatrix} &= \left(\frac{1}{M+1}\underbrace{\sum_{i=0}^{M} p_{ij}}_{=1}, \ldots, \frac{1}{M+1}\underbrace{\sum_{i=0}^{M} p_{iM}}_{=1}\right) = \\
&= \left(\frac{1}{M+1}, \ldots, \frac{1}{M+1}\right)
\end{aligned}$$

So $\pi = (1/(M+1), \ldots, 1/(M+1))$ solves the system and since the solution is unique, π is the solution we wanted!

Exercise 4.22

Indication: Consider a new process X_n defined as $X_n := Y_n \mod 13$ (This means, X_n is the remainder of the integer division of Y_n/X_n, if such remainder is 0, this means that Y_n is a multiple of 13). Observe that the state space of Y is $S = \{0, 1, 2, \ldots, 12\}$ and that the number of multiples of 13 in X_n is the same as the number of 0's in Y_n.

Finally, the transition probabilities of Y are easy to compute. (To conclude, use the previous exercise to get $\pi = (1/13, \ldots, 1/13)$).