STK2130: Solution to Exam Spring 2012

Problem 1

(a) Define what you mean by a branching process.

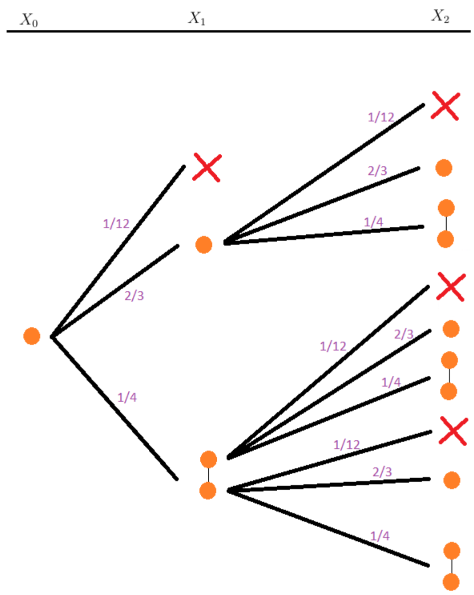
See page 245-246, Section 4.5 (Branching processes)

(b) Assume that a branching process $\{X_n\}$ starts with one individual ($X_0 = 1$) and has an offspring distribution given by

$$P(\xi = 0) = \frac{1}{12}, \; P(\xi = 1) = \frac{2}{3}, \; P(\xi = 2) = \frac{1}{4}.$$

Find the probability distribution of X_2.

To solve this problem it is highly recommended to plot a Venn's diagram of the situation. See below.



Then we see how to compute all probabilities in an easy way since each step is independent of the rest. So

$$P(X_2 = 0) = \frac{1}{12} + \frac{2}{3}\frac{1}{12} + \frac{1}{4}\frac{1}{12}\frac{1}{12} = \frac{9}{64},$$

$$P(X_2 = 1) = \frac{2}{3}\frac{2}{3} + \frac{1}{4}\left(\frac{1}{12}\frac{2}{3} + \frac{2}{3}\frac{1}{12}\right) = \frac{17}{36},$$

$$P(X_2 = 2) = \frac{2}{3}\frac{1}{4} + \frac{1}{4}\left(\frac{1}{12}\frac{1}{4} + \frac{2}{3}\frac{2}{3} + \frac{1}{4}\frac{1}{12}\right) = \frac{83}{288},$$

$$P(X_2 = 3) = \frac{1}{4}\left(\frac{2}{3}\frac{1}{4} + \frac{1}{4}\frac{2}{3}\right) = \frac{1}{12},$$

$$P(X_2 = 4) = \frac{1}{4}\frac{1}{4}\frac{1}{4} = \frac{1}{64}.$$

Observe or check that the sum of all is indeed 1.

(c) Find the probability of ultimate extinction for this process. We see that

$$\mu = \sum_j jP(\xi = j) = \frac{2}{3} + 2\frac{1}{4} = \frac{2}{3} + \frac{1}{2} = \frac{7}{6} > 1.$$

If we denote by π_0 the probability that the population dies out/ultimate extinction. Then

$$\pi_0 = 1 \quad \text{if} \quad \mu \leqslant 1$$

$$\pi_0 < 1 \quad \text{if} \quad \mu > 1.$$

We are in the second situation. To exactly determine π_0 we have to solve the following equation

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P(\xi = j)$$

(See page 248) and choose the smallest positive root. Thus

$$\pi_0 = \frac{1}{12} + \frac{2}{3}\pi_0 + \frac{1}{4}\pi_0^2$$

which has solution $\pi_0 = 1/3$.

Problem 2

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.2 & 0.3 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.2 & 0.3 & 0 \\ 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

(a) There are three classes $\{1\}$ (recurrent), $\{2,3\}$ (transient) and $\{4,5\}$ (recurrent)

(b) Given we are in state 2, probability of ultimate absorption in state 1:

We aim at computing the probability of absorption (in this case in 1). So, we can consider the class $\{4,5\}$ as a single state which is absorbing. So rearranging the matrix P, we are interested only in a process $\tilde{X}_n$ having transition probability matrx $\tilde{P}$ as follows:

$$\tilde{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 0.3 & 0.5 & 0 \\ 0 & 0.5 & 0.2 & 0.3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In general, for an absorbing state i, denote by $N_i = \min\{n \geqslant 0 : X_n = i\}$. Then observe that

$$P(N_i < \infty | X_0 = i) = 1$$

$$P(N_j < \infty | X_0 = i) = 0 \text{ if } i \neq j \text{ are absorbing states}$$

$$P(N_j < \infty | X_0 = i) = \sum_{k \in S} P(N_j < \infty | X_1 = k) p_{ik} \text{ if } i \text{ is transient.}$$

The proof of the above summation:

$$\begin{aligned} P(N_j < \infty | X_0 = i) &= \sum_{k \in S} P(N_j < \infty, X_1 = k | X_0 = i) \\ &= \sum_{k \in S} \frac{P(N_j < \infty, X_1 = k, X_0 = i)}{P(X_0 = i)} \frac{P(X_1 = k, X_0 = i)}{P(X_1 = k, X_0 = i)} \\ &= \sum_{k \in S} P(N_j < \infty | X_1 = k, X_0 = i) P(X_1 = k | X_0 = i) \\ &= \sum_{k \in S} q_k p_{ik} \end{aligned}$$

In our case $j = 1$, writing down the equation for $X_0 = 2$ we get,

$$q_2 = \tilde{p}_{21} + q_2 \tilde{p}_{22} + q_3 \tilde{p}_{23}$$

We do not have enough equations so we use the ones for q_3 as well,

$$q_3 = \tilde{p}_{31} + q_2 \tilde{p}_{32} + q_3 \tilde{p}_{33}.$$

Solutions are: $q_2 = 0.5161$ and $q_3 = 0.32258$ so

$$P(N_1 < \infty | X_0 = 2) = 0.5161.$$

(c) Starting at 2, find the expected time until entering one of the recurrent states.

Denote by $T = \{\text{time to absorption}\}$ and, similarly as before, for a transient state i denote $\mu_i = E[T | X_0 = i]$ and $\mu = E[T]$. Observe first that, if we start at a transient state we need at least 1 time period, so

$$E[T | X_0 = i] = 1 + E[T],$$

which can be computed using conditional expectations and the tower property via a system of equations

$$\mu_2 = 1 + \mu_2 \tilde{p}_{22} + \mu_3 \tilde{p}_{23}$$

$$\mu_3 = 1 + \mu_3 \tilde{p}_{32} + \mu_3 \tilde{p}_{33}$$

Moreover, observe that $\mu_1 = \mu_4 = 0$. Solutions are: $\mu_2 = 4.1935$ and $\mu_3 = 3.871$. In matrix form, these equations can be solved as follows: Take the transient states $\{2, 3\}$ and denote by P_T the matrix corresponding to these states:

$$P_T = \begin{pmatrix} 0.3 & 0.5 \\ 0.5 & 0.2 \end{pmatrix}$$

Then $S = (I - P_T)^{-1}$ is a matrix with entries s_{ij}, i, j transient, meaning:

$$s_{ij} = \text{"The expected number of periods that we are in j starting from i"}$$

Hence, $s_{22} + s_{23}$ are the expected number of periods to go to 2 and 3 starting from 3 and since there are no more transient states, these are all expected time periods in the transient class (i.e. before exit). In our case:

$$S = \begin{pmatrix} 2.5806 & 1.6129 \\ 1.6129 & 2.2581 \end{pmatrix}$$

and $\mu_2 = s_{22} + s_{23} = 4.1935$.

(d) Knowing we have entered one of the states 4 or 5. Find the stationary distribution of these two states.

Since this class is closed, we will not get out and we have that the transition probability matrix of this class is then

$$\begin{pmatrix} 0.4 & 0.6 \\ 0.5 & 0.5 \end{pmatrix}$$

The asymptotic distribution is defined as $\pi_i = \lim_{n\to\infty} p_{ij}^n$ whenever this limit exists for each $i \in S$. In our case we want to find the vector of probabilities $\pi = (\pi_4, \pi_5)$ which can be found as the solution to $\pi = \pi P$ and the fact that $\pi_4 + \pi_5 = 1$. Thus

$$\pi = (45/99, 54/99)$$

i.e. the proportion of times we are in state 4 is 4/9 and the proportion of times we are in 5 is 5/9.

Problem 3

Consider a two-dimensional Poisson process of particles in the plane with intensity parameter λ.

(a) What is the expectation and the variance of the number of particles in a disc of radius r?

A two-dimensional Poisson process is a process $\{N(B), B \subset \mathbb{R}^2\}$ where

$$N(B) := \{\text{number of events in } B\}$$

has distribution function

$$P(N(B) = k) = \frac{e^{-\lambda|B|}(\lambda|B|)^k}{k!}$$

where $|B|$ means the area of the subset B. So, the above gives us the probability that k events occur inside the set B. Thus, denoting $B((x, y), r)$ the ball of center $(x, y) \in \mathbb{R}^2$ radius r we have that

$$E[N(B((x, y), r))] = Var[N(B((x, y), r))] = |B((x, y), r)|\lambda = \pi r^2 \lambda.$$

(b) Determine the distribution function of the distance D between a particle and its nearest neighbor.

Let $D = dist(x, y)$ where y is the nearest neighbor of x, this means that inside the circle of center x and radius D there are no events because y is the closest one. Without loss of generality let us assume that $x = (0, 0)$ the origin. Thus denoting $F(d) = P(D < d)$, for $d \geqslant 0$, the distribution function of D we have

$$F(d) = P(D \leqslant d) = 1 - P(D > d) = 1 - P(N(B((0,0),d)) = 0) = 1 - e^{-\pi\lambda d^2}.$$

(which satisfies the properties of a distribution function, indeed)

(c) Compute the expectation of the distance.

The density function is $f(d) = F'(d) = 2\pi\lambda d e^{-\pi\lambda d^2}\mathbf{1}_{\{d\geqslant 0\}}$. So (using integration by parts and the change of variables $x^2 = \pi\lambda d^2$) we obtain

$$\begin{aligned} E[D] &= 2\pi\lambda \int_0^\infty d^2 e^{-\pi\lambda d^2}\mathbf{d}d = \int_0^\infty e^{-\pi\lambda d^2}\mathbf{d}d \\ &= \frac{1}{\sqrt{\pi\lambda}}\int_0^\infty e^{-x^2}\mathbf{d}x = \frac{1}{2\sqrt{\lambda}}. \end{aligned}$$