Exercise 6.11

(a) Denote by

$$T_i = \{ \text{ time to make a transition from } i \text{ to } i+1\}$$

Given we are in state i, at a given time $X(t) = i$, then each individual (independent of the rest) will give birth with rate λ, so altogether, the rate of change is λ for each of them, as a consequence

$$T_i \sim \exp(i\lambda).$$

(b) The maximum of X_i, $i = 1, \ldots j$ has the same distribution as $\varepsilon_1 + \cdots + \varepsilon_j$, where $\varepsilon_i = \{\text{time from the } i-1\text{-th to the } i\text{-th failure}\}$ since $\max\{X_1, \ldots, X_j\}$ is the time to absolute failure which is the intermediate times between each failure. Now, for example, ε_1 is the time to the first failure, so

$$\varepsilon = \min\{X_1, \ldots, X_n\}$$

with distribution

$$P(\min\{X_1, \ldots, X_n\} > t) = P(X_1 > t)^j = e^{-j\lambda t} \Rightarrow \varepsilon_1 \sim \exp(j\lambda).$$

For ε_2 we have

$$P(\varepsilon_2 > t) = P(\min\{X_2, \ldots, X_j\} > t) = P(X_1 > t)^{j-1} = e^{-(j-1)\lambda t} \Rightarrow \varepsilon_2 \sim \exp((j-1)\lambda).$$

For an aribitrary ε_i, $i = 1, \ldots, j$ we have

$$P(\varepsilon_i > t) = P(\min\{X_i, \ldots, X_j\} > t) = P(X_1 > t)^{j-i+1} = e^{-(j-i+1)\lambda t} \Rightarrow \varepsilon_i \sim \exp((j-i+1)\lambda).$$

(c)

$$P(T_1 + \cdots + T_j \leqslant t) = P(\max\{X_1, \ldots, X_j\} < t) = (1 - e^{-\lambda t})^j$$

(d)

$$\begin{aligned} P_{1j}(t) &= P(X(t) = j|X(0) = 1) = P(X(t) \geqslant j|X(0) = 1) - P(X(t) \geqslant j+1|X(0) = 1) \\ &= P(T_1 + \cdots + T_{j-1} \leqslant t) - P(T_1 + \ldots T_j \leqslant t) \\ &= e^{-\lambda t}(1 - e^{-\lambda t})^{j-1}. \end{aligned}$$

We see directly that $X(t) = j|X(0) = 1 \sim \text{Geom}(e^{-\lambda t})$ which is also natural.

(e) The sum of i independent geometric r.v., each having parameter $p = e^{-\lambda t}$ is a negative binomial with parameters i, p (with consistent definitions of each one, see the **Note** below). The result follows since starting with an initial population of i is equivalent to having i independent Yule processes, each starting with a single individual, i.e.

$$(X(t) = j|X(0) = i) = \sum_{k=1}^{i}(X(t) = j|X(0) = 1) \sim \sum_{k=1}^{i} \text{Geom}(e^{-\lambda t}) = i\text{Geom}(e^{-\lambda t}) = \text{NB}(i, e^{-\lambda t}).$$

For skeptical students, to prove that the sum of random variables has some specific distribution, the easiest way is by computing their characteristic function since

$$\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t)$$

where $\varphi_Z(t)$ denotes the characteristic function of a r.v. Z. The characteristic function of a r.v. Z characterizes the distribution of Z, that is

$$X \text{ and } Y \text{ have the same distribution} \iff \varphi_X = \varphi_Y.$$

The caracteristic function of a geometric r.v. with parameter p is:

$$\varphi_{geom(p)}(t) = \frac{pe^{it}}{1-(1-p)e^{it}}.$$

Then we know that the characteristic function of n geometric r.v. is:

$$\varphi_{n \cdot geom(p)}(t) = \left(\frac{p}{1-(1-p)e^{it}}\right)^n$$

which corresponds to the characteristic function of a negative binomial r.v. with parameters n, p.

Note: One has to be careful when computing the characteristic functions, we have to make sure that both r.v. (that we wish to compare) have the same support. Moreover, depending on the definition of the r.v. one might get $Neg.Bin(n, 1-p)$ instead of $Neg.Bin(n, p)$.