UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK2130 — Modelling by Stochastic Processes: BRIEF OUTLINE OF SOLUTIONS

Day of examination: Friday 17 June 2016

This problem set consists of 3 pages.

Appendices: None

Permitted aids: Approved calculator. "Formelsamling til STK1100 og STK1110"

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a

Because all rows in a transition probability matrix must sum to 1, we have $p = 0.2$ and $q = 1$

b

The only states that communicate are 2 and 3, hence we have three classes: $\{0\}$, $\{1\}$ and $\{2,3\}$. $\{0\}$ and $\{1\}$ are transient and $\{2,3\}$ is recurrent.

c

The recurrent class $\{2,3\}$ is irreducible, positive recurrent and aperiodic. Hence for $j = 2,3$ the limiting probabilities $\lim_{n\to\infty} P_{ij}^n$ exist and equal the stationary probabilities π_j. They can be found by solving $\pi_2 = 0.3\pi_3$ and $\pi_2 + \pi_3 = 1$. Hence, $\pi_2 = \frac{3}{13}$, $\pi_3 = \frac{10}{13}$.

d

"One-step-ahead" analysis:

$$\nu_1 = 0.2(\nu_1 + 1) + 0.8 \Rightarrow \nu_1 = 1.25$$
$$\nu_0 = 0.6(\nu_0 + 1) + 0.2 * (\nu_1 + 1) + 0.2 \Rightarrow \nu_0 = 3.125$$

Problem 2

a

There can be a minimum of 0 customers in the system, or a maximum of $N+1$ customers in the system (one being served plus N in the queue). Hence

(Continued on page 2.)

$S = \{0, 1, \ldots, N+1\}$. Transitions from state i may go only to either state $i+1$ (for $0 \le i < N+1$) or $i-1$ (for $0 < i \le N+1$), hence it is a birth and death process. When there are $i = 0, \ldots, N$ customers in the system, a new arriving customer will go directly into service if $i = 0$, or otherwise join the queue, and the arrivals follow a Poisson process with rate λ. Hence, $\lambda_i = \lambda, i = 0, \ldots, N$. Otherwise, $\lambda_i = 0$. When there are $i = 1, \ldots, N+1$ customers in the system, one customer is being served and the service time is exponential with rate μ, hence $\mu_i = \mu, i = 1, \ldots, N+1$. Obviously $\mu_0 = 0$.

b

$$\begin{aligned} \lambda P_0 &= \mu P_1 \\ (\lambda + \mu) P_i &= \lambda P_{i-1} + \mu P_{i+1}, 1 \le i \le N \\ \mu P_{N+1} &= \lambda P_N \end{aligned}$$

c

R is the $(N+2) \times (N+2)$ matrix, where row $i+1$ represents state i, and column $j+1$ represents state j, with elements

$$\begin{aligned} &r_{11} = -\lambda,\ r_{12} = \lambda, r_{1j} = 0,\ j \notin \{1, 2\} \\ &r_{N+2,N+1} = \mu,\ r_{N+2,N+2} = -\mu, r_{N+2,j} = 0, j \notin \{N+1, N+2\} \\ &r_{i,i-1} = \mu,\ r_{i,i} = -(\lambda + \mu),\ r_{i,i+1} = \lambda,\ r_{ij} = 0,\ 1 < i < N+2, j \notin \{i-1, i, i+1\} \end{aligned}$$

Alternatively, the matrix can be given in this way

$$R = \begin{pmatrix} -\lambda & \lambda & 0 & \cdots & & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 & \cdots & 0 \\ 0 & \mu & -(\lambda+\mu) & \lambda & 0 & \cdots \\ \vdots & & & \vdots & & \\ 0 & \cdots & & 0 & \mu & -\mu \end{pmatrix}$$

d

$$P(t) = U e^{Lt} U^{-1} = \begin{pmatrix} 1 & -\lambda \\ 1 & \mu \end{pmatrix} \begin{pmatrix} e^0 & 0 \\ 0 & e^{-(\lambda+\mu)t} \end{pmatrix} \frac{1}{\lambda + \mu} \begin{pmatrix} \mu & \lambda \\ -1 & 1 \end{pmatrix}$$

which means that

$$\begin{aligned} \lim_{t \to \infty} P(t) &= \frac{1}{\lambda + \mu} \begin{pmatrix} 1 & -\lambda \\ 1 & \mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mu & \lambda \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{\lambda + \mu} \begin{pmatrix} \mu & \lambda \\ \mu & \lambda \end{pmatrix} \end{aligned}$$

The distribution defined by P_0 and P_1 is the Bernoulli distribution with parameter $p = \frac{\lambda}{\lambda + \mu}$.

(Continued on page 3.)

Problem 3

a

Obviously $B(0) = 0$. Since all the increments $X(t) - X(s)$ are stationary and independent, all the increments $B(t) - B(s)$ are also stationary and independent. $\mathrm{E}[B(t)] = \frac{\mu t - \mu t}{\sigma} = 0$ and $\mathrm{Var}[B(t)] = \frac{\sigma^2 t}{\sigma^2} = t$, hence $B(t) - B(s) \sim N(0, t - s)$, $0 \leq s < t$.

b

We can write $B(s) + B(t) = 2B(s) + B(t) - B(s)$. We have $2B(s) \sim N(0, 4s)$ and $B(t) - B(s) \sim N(0, t - s)$. Since $B(s)$ and $B(t) - B(s)$ are independent (because of independent increments), we then get $B(s) + B(t) \sim N(0, 3s + t)$.