STK2130 Spring 2016 – Mandatory assignment

Deadline Thursday March 31st, 14:30

You are allowed to collaborate and discuss the problems with other students, but each student has to formulate her or his own answers. You should give the names of the students you collaborate with, so that it is possible to compare the written solutions.

The assignment consists of 3 Problems over 3 pages. Make sure you have the complete assignment.

The answer to the exam project should be handed in on paper on the 7th floor in NHA (Niels Henrik Abels hus). You may deliver a handwritten or Latex/Word-processed answer to the project in English or Norwegian.

Problem 1

A Markov chain $X_0, X_1, X_2, \ldots$ on the states $\{0, 1, 2, 3, 4\}$ is defined by the transition matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

a) The chain has three classes, $\mathcal{C}_0 = \{0, 1\}, \mathcal{C}_1 = \{2, 3\}, \mathcal{C}_2 = \{4\}$ where $\mathcal{C}_0$ is transient, $\mathcal{C}_1$ is closed and $\mathcal{C}_2$ is absorbing (and closed). Explain why this is so.

Which of the classes are recurrent?

b) Let T be the time until the chain enters one of the closed classes and define $\mu_i = \mathrm{E}[T|X_0 = i]$ for $i \in \mathcal{C}_0$.

Explain why the μ_i satisfy the following two equations

$$(1) \quad \mu_0 = (\mu_0 + 1)\tfrac{1}{5} + (\mu_1 + 1)\tfrac{1}{5} + \tfrac{3}{5}$$

$$(2) \quad \mu_1 = (\mu_0 + 1)\tfrac{2}{5} + (\mu_1 + 1)\tfrac{1}{5} + \tfrac{2}{5}$$

Solve the equations (1) and (2) to obtain μ_0 and μ_1.

c) Let q_i be the probability that the chain ends up in state 4 conditional on $X_0 = i$. Find and explain equations for obtaining q_0 and q_1. Solve the equations.

d) Let s_{ij} denote the expected number of visits to states $j = 0$ and $j = 1$ conditional on $X_0 = i$. Find and solve equations for determining the s_{ij}.

Problem 2

Let a Markov chain $X_0, X_1, X_2, \ldots$ on the states $\{0, 1, 2, 3, 4\}$ be described by the transition matrix

$$\mathbf{P} = \begin{bmatrix} q & p & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ q & 0 & 0 & p & 0 \\ q & 0 & 0 & 0 & p \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $0 < p < 1$ and $q = 1 - p$.

a) Explain why this Markov chain has a stationary distribution with limit probabilities $\pi_j = \lim_{n\to\infty} \mathrm{P}(X_n = j | X_0 = i)$ determined by equations

$$(1) \quad \pi_j = p\pi_{j-1} \ \text{ for } j = 1, 2, 3, 4$$

$$(2) \quad \textstyle\sum_{j=0}^{4} \pi_j = 1$$

b) Show that the solution to the equations in (a) are given by

$$\pi_j = \frac{1-p}{1-p^5} p^j \ \text{ for } j = 0, 1, 2, 3, 4$$

c) Let V_n^j be the indicator that $X_n = j$ (i.e. $V_n^j = 1$ if $X_n = j$ and $V_n^j = 0$ otherwise).

What is the limit of $\bar{V}_n^j = \frac{1}{n}\sum_{i=1}^{n} V_i^j$? when $n \to \infty$?

(d) For some function $h()$ state the limit of $\frac{1}{n}\sum_{i=1}^{n} h(X_i)$ when $n \to \infty$?

Evaluate the limits numerically for (i) $h(x) = x$ and (ii) $h(x) = \exp(x)$ when $p = 0.3$.

Problem 3

For this problem it is recommended to use some software like `R`, MATLAB or Python.

We will consider the same Markov chain as in Problem 2 using $p = 0.3$ (and $q = 0.7$).

a) Let $\mathbf{P}^n$ be the product of the transition matrix $\mathbf{P}$ by itself n times. What does the elements P_{ij}^n of $\mathbf{P}^n$ signify?

   Calculate $\mathbf{P}^n$ for $n = 2, 4$ and 8. Compare with the limiting distribution that you derived in question (b) in Problem 2.

b) Sample and plot a sequence $X_0, X_1, \ldots, X_{50}$ from the Markov chain with start value $X_0 = 0$.

c) Run sequences $X_0, X_1, \ldots, X_n$ for $n = 1000$ and $n = 10000$. Calculate the proportion of time that the chain attains different values. Compare with Problem 2, question (c).

   Then calculate $\frac{1}{n}\sum_{i=1}^{n} h(X_i)$ for (i) $h(x) = x$ and (ii) $h(x) = \exp(x)$. Compare with the results you obtained in question (d) of Problem 2.