UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK2130 — Modelling by Stochastic Processes

Day of examination: Friday 17 June 2016

Examination hours: 09.00 – 13.00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: Approved calculator. "Formelsamling til STK1100 og STK1110"

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Consider a discrete time Markov Chain with state space $\{0, 1, 2, 3\}$. The matrix of one-step transition probabilities is

$$P = \begin{pmatrix} 0.6 & p & 0 & p \\ 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & q \\ 0 & 0 & 0.3 & 0.7 \end{pmatrix}$$

a

Determine p and q.

b

Determine the classes of the Markov Chain. For each class, determine whether it is transient or recurrent.

c

Argue that the limits $\lim_{n\to\infty} P_{ij}^n$ exist for $j = 2$ and $j = 3$. Find these limits.

d

Let T be the time until the chain reaches a recurrent state. Find $\nu_i = \mathrm{E}[T \mid X(0) = i]$ for all transient states i.

Problem 2

Consider a service counter with only one server. Customers arrive according to a Poisson process with rate λ. If the server is free, the arriving customer

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get served immediately. If the server is occupied, and there are less than N people waiting in line, the arriving customer joins the queue. If however there are N people waiting in line, then an arriving customer leaves without joining the queue. When the server finishes serving a customer, the customer leaves and if there is a queue, the next customer in line gets served. The service times are assumed to be independent, exponential random variables with rate μ. Let $X(t)$ be the number of customers in the system at time t. You can assume that $\{X(t), t \geq 0\}$ is a continuous time Markov Chain with homogeneous transition probabilities

$$P_{ij}(t) = P(X(t) = j \mid X(0) = i) = P(X(t+s) = j \mid X(s) = i)$$

a

Give an intuitive explanation for this being a birth and death process with state space $\{0, 1, \ldots, N+1\}$ and birth and death rates

$$\begin{aligned} \mu_0 &= 0 \\ \mu_i &= \mu, \quad 1 \leq i \leq N+1 \\ \lambda_i &= \lambda, \quad 0 \leq i \leq N \\ \lambda_{N+1} &= 0 \end{aligned}$$

b

Let $P_j \equiv \lim_{t\to\infty} P_{ij}(t)$. Put up the set of balance equations that can be solved to find $P_j, j = 0, \ldots, N+1$, without solving them.

c

Put up the infinitesimal generator matrix R.

d

Consider now the special case that $N = 0$, hence if the server is busy, arriving customers will not wait in line, but just leave immediately. The solution to the Kolmogorov backward and forward equations in matrix form is

$$P(t) = e^{Rt} = \sum_{n=0}^{\infty} R^n \frac{t^n}{n!}$$

It can be shown that we can write $R = ULU^{-1}$, where

$$L = \begin{pmatrix} 0 & 0 \\ 0 & -(\lambda+\mu) \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & -\lambda \\ 1 & \mu \end{pmatrix},$$

$$U^{-1} = \frac{1}{\lambda+\mu} \begin{pmatrix} \mu & \lambda \\ -1 & 1 \end{pmatrix},$$

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and hence

$$P(t) = e^{Rt} = Ue^{Lt}U^{-1},$$

where

$$e^{Lt} = \begin{pmatrix} e^0 & 0 \\ 0 & e^{-(\lambda+\mu)t} \end{pmatrix}.$$

Find $\lim_{t\to\infty} P(t)$ expressed by λ and μ. What do we call this distribution defined by P_0 and P_1?

Problem 3

Let $\{X(t), t \geq 0\}$ be a Brownian motion with drift coefficient μ and variance parameter σ^2, which means that (i) $X(0) = 0$, (ii) all the increments $X(t) - X(s)$ are stationary and independent and (iii) $X(t) - X(s) \sim N(\mu(t-s), \sigma^2(t-s))$, $0 \leq s < t$.

a

Show that

$$B(t) = \frac{X(t) - \mu t}{\sigma}$$

is a standard Brownian motion.

b

What is the distribution of $B(s) + B(t), s \leq t$?

THE END