Theorem 5.3 If $\{N(t), t \geq 0\}$ is a nonstationary Poisson process with intensity function $\lambda(t)$, $t \geq 0$, then $N(t+s) - N(s)$ is a Poisson random variable with mean $m(t+s) - m(s) = \int_s^{t+s} \lambda(y)dy$

$$m(t) = \int_0^t \lambda(y)dy$$

Proof

1^{st} step: show that $N(t) \sim \text{Poisson}(m(t))$

From $g(t) = E[e^{-uN(t)}]$ (Laplace transform, see also proof of Theorem 5.1)

$$g(t+h) = g(t) E[e^{-uN_t(h)}]$$

where $N_t(h) = N(t+h) - N(t)$

From axioms (iii) and (iv) of Definition 5.3 $\to$ $\begin{cases} P[N_t(h)=1] = \lambda(t)h + o(h) \\ P[N_t(h) \geq 2] = o(h) \\ \Rightarrow P[N_t(h)=0] = 1 - \lambda(t)h + o(h) \end{cases}$

Therefore

$$g(t+h) = g(t)\Big[\underbrace{1 - \lambda(t)h}_{e^{-u0}\cdot P[N_t(h)=0]} + \underbrace{e^{-u}\lambda(t)h}_{e^{-u\cdot 1}\cdot P[N_t(h)=1]} + \underbrace{o(h)}_{o(h) \,\geq\, P[N_t(t)\geq 2]}\Big]$$

$$g(t+h) - g(t) = \lambda(t)h(e^{-u} - 1)g(t) + o(h)g(t)$$

Dividing by h and letting $h \to 0$

$$\underbrace{\frac{g(t+h) - g(t)}{h}}_{g'(t)} = \frac{\lambda(t)\cancel{h}(e^{-u} - 1)g(t)}{\cancel{h}} + \underbrace{\frac{o(h)}{h}}_{\to 0}$$

$$\frac{g'(t)}{g(t)} = \lambda(t)(e^{-u} - 1)$$

Integrating from 0 to t both sides

$$\int_0^t \frac{g'(t)}{g(t)}dt = \int_0^t \lambda(t)(e^{-u} - 1)dt$$

$$\log(g(t)) - \log(g(0)) = (e^{-u} - 1)\int_0^t \lambda(t)dt$$

Since $g(0) = e^{-0\cancel{\cdot 0}} = 1$ and $\int_0^t \lambda(t)dt = m(t)$,

$g(t) = \exp\{(e^{-u} - 1)m(t)\}$ $\Rightarrow$ Laplace transform of $N(t)$ is equal to the Laplace transform of a Poisson r.v.

$\Downarrow$

$N(t)$ is distributed like a Poisson$(m(t))$

$N_s(t) = N(t+s) - N(s)$, $\{N_s(t), t\geq 0\}$ is a nonstationary Poisson process with intensity function $\lambda_s(t) = \lambda(s+t)$, $t>0$. Therefore, $N_s(t)$ is distributed like a Poisson random variable with mean

$$\int_0^t \lambda_s(y)\,dy = \int_0^t \lambda(s+y)\,dy = \int_s^{s+t} \lambda(t)\,dt \qquad \text{q.e.d.}$$

Example of last lecture

$\{N(t), t\geq 0\}$ a Poisson process

$N_c(t)$ counts the number of events which are randomly picked from the events of the starting Poisson process with probability $p(t)$

$\Rightarrow \{N_c(t), t\geq 0\}$ is a nonstationary Poisson process with intensity function $\lambda(t) = \lambda p(t)$.

We show that $\{N_c(t), t\geq 0\}$ satisfies the axioms of Definition 5.3

(i) $N_c(t) = 0$ follow from properties of $\{N(t), t\geq 0\}$

(ii) $\{N_c(t), t\geq 0\}$ has independent increments, follow from property of $\{N(t), t\geq 0\}$

(iii) $\mathbb{P}[N_c(t+h) - N_c(t) \geq 2) = o(h)$

$$\leq \mathbb{P}[N(t+h) - N(t) \geq 2] = o(h)$$

(iv) $\mathbb{P}[\overbrace{N_c(t+h) - N_c(t)}^{N_c(t,t+h)} = 1] = \mathbb{P}[N_c(t,t+h) = 1]$

$p(t)\lambda h + o(h)$

$$\overset{!}{=} \overbrace{\mathbb{P}[N_c(t,t+h)=1 \mid N(t,t+h)=1]}^{p(t)}\,\overbrace{\mathbb{P}[N(t,t+h)=1]}^{\lambda h + o(h)} +$$

$$+\,\mathbb{P}[N_c(t,t+h)=1 \mid N(t,t+h)\geq 2]\,\underbrace{\mathbb{P}[N(t,t+h)\geq 2]}_{o(h)}$$

$$= p(t)\lambda h + o(h)$$

The importance of a nonstationary Poisson process resides in the fact that we do not longer require stationary increments but only independent increments.

E.g.: the number of customer who enter in a shop can be modelled by ordinary Poisson process if the shop does not experience rush hours. With a nonstationary Poisson process we can model the process even if the shop experiences rush hours (given independence of arrival in nonoverlapping time intervals)

Example 5.24

A hot-dog stand open at 8am.

From 8am to 11am customers arrival average rate is steadly increasing from 5 per hour (8am) to 20 per hour (11am)

From 11am to 1pm the average rate stays at 20 per hour

From 1pm to 5pm the average arrival rate steadly decreases from 20 per hour (1pm) to 12 per hour (5pm)

At 5pm the stand closes.

Assuming that the number of customers arriving during disjoint time intervals are independent:

(a) what is a good probability model for the process

(b) which is the probability of no customer between 8:30 am to 9:30 am on Monday

(c) which is the expected number of customers in this $\overbrace{\text{time interval}}^{?}$

Sol.

(a) a good model is a nonstationary Poisson process with intensity function

$$\lambda(t) = \begin{cases} 5+5t & 0 \le t \le 3 \\ 20 & 3 < t \le 5 \\ 20-2(t-5) & 5 < t \le 9 \end{cases} \qquad \lambda(t-9) = \lambda(t) \quad \text{for } t>9$$

if we only consider the opening hours. Otherwise

$$\lambda(t) = \begin{cases} 0 & 0 \le t < 8 \\ 5+5(t-8) & 8 \le t < 11 \\ 20 & 11 \le t < 13 \\ 20-2(t-13) & 13 \le t \le 17 \\ 0 & 17 < t \le 24 \end{cases} \qquad \lambda(t-24) = \lambda(t) \quad \text{for } t>24$$

(b) the number of arrivals between 8:30 am and 9:30 $\sim$ Poisson$(m(\frac{3}{2})-m(\frac{1}{2}))$

$\mathbb{P}$[no customers in this time interval] $= \exp\{-\int_{1/2}^{3/2}(5+5t)\,dt\} = e^{-10}$

$\lambda^k e^{-\lambda}/k!$

(c) average number of arrivals $\int_{1/2}^{3/2}(5+5t)\,dt = 10$

$$= \left[5t+\frac{5}{2}t^2\right]_{1/2}^{3/2} = \frac{15}{2}+\frac{45}{8}-\frac{5}{2}-\frac{5}{8} = \frac{60+45-20-5}{8} = \frac{80}{8} = 10$$

Remark 1: Proposition 5.2 covering ordinary Poisson processes can be extended for the nonstationary case
Suppose that events occur accordingly to a Poisson process with rate λ, and, independently of what occurred before, at time s an event can be of type 1 with probability $p_1(s)$ or of type 2 with probability $p_2(s) = 1 - p_1(s)$. If we denote with $N_i(t), t \geq 0$ the number of events of type i, then by Definition 5.3, $\{N_i(t), t \geq 0\}$ is a nonstationary Poisson process with intensity function

$$\lambda_i(t) = \lambda\, p_i(t) \quad , \text{ respectively}$$

(proof is the same of what we did for example at the beginning of the lecture)
(please check that this result confirm what we saw in Proposition 5.3)

Remark 2: If we denote with S_n the time of the n-th event of the nonstationary Poisson process, then its density is

$$\mathbb{P}[t \leq S_n \leq t+h] = \mathbb{P}[N(t) = n-1, \text{ one event occur in } (t;t+h)] + o(h)$$

independent increments

$$= \mathbb{P}[N(t) = n-1]\, \mathbb{P}[\text{one event occur in } (t;t+h)] + o(h)$$

Definition 5.3 (iv)

$$= e^{-m(t)} \frac{m(t)^{n-1}}{(n-1)!} [\lambda(t) h + o(h)] + o(h)$$

$$= \lambda(t) e^{-m(t)} \frac{m(t)^{n-1}}{(n-1)!} h + o(h)$$

Dividing by h and letting $h \to 0$

$$f_{S_n}(t) = \lambda(t) e^{-m(t)} \frac{m(t)^{n-1}}{(n-1)!} + \underbrace{\frac{o(h)}{h}}_{\xrightarrow{h\to 0} 0} \qquad \text{where } m(t) = \int_0^t \lambda(y)\,dy$$

5.4.2 Compound Poisson Process

Definition: a stochastic process $\{X(t), t \geq 0\}$ is said to be a compound Poisson process (random variable) if it can represented as

$$X(t) = \sum_{i=1}^{N(t)} Y_i \quad , \quad t \geq 0$$

Note
$E[X(t)] = \lambda t E[Y_i]$
$Var[X(t)] = \lambda t E[Y_i^2]$ (5.32)

where:
- $\{N(t), t \geq 0\}$ is a Poisson process
- $\{Y_i, i \geq 1\}$ is a family of i.i.d. random variables that is also independent of $\{N(t), t \geq 0\}$

Examples of compound Poisson processes

(ii) Busses arrive to a sport event following a Poisson process $\{N(t), t \geq 0\}$
The number of fans in each bus is iid. Y_i $Y_i :=$ # of fans in bus i
Then $\{X(t), t \geq 0\}$ is a compound Poisson process | the number of fans who arrive at the sport event $X(t)$ is

(iii) Suppose customers leave a supermarket following a Poisson process
$N(t) :=$ count of customers who leave the supermarket
$Y_i :=$ amount of money spent by customer i (suppose iid)
$X(t)$ counts the amount of money spent by time t
$\{X(t), t \geq 0\}$ is a compound Poisson process

Example 5.26
Suppose that families migrate to a certain area at a Poisson rate $\lambda = 2$ per week.
The family size is iid

$$Y_i = \begin{cases} 1 & \text{with prob } 1/6 \\ 2 & \text{" " } 1/3 \\ 3 & \text{" " } 1/3 \\ 4 & \text{" " } 1/6 \end{cases}$$

(a) which is the expecting number of individuals arriving to the area in a 5-week period
(b) which is the variance

$\{N(t), t \geq 0\}$ is a Poisson process (at the level of families)
$\{X(t), t \geq 0\}$ is a compound Poisson process (at the level of individuals)
Y_i is the number of individuals of family i

$$E[Y_i] = 1 \times 1/6 + 2 \times 1/3 + 3 \times 1/3 + 4 \times 1/6 = \frac{1+4+6+4}{6} = \frac{15}{6} = \frac{5}{2} \; *$$

$$E[Y_i^2] = 1^2 \times 1/6 + 2^2 \times 1/3 + 3^2 \times 1/3 + 4^2 \times 1/6 = \frac{1+8+18+16}{6} = \frac{43}{6}$$

$X(t)$ is the number of individuals

(a) $E[X(5)] = \underbrace{\lambda}_{2 \text{ per week}} \underbrace{t}_{5 \text{ weeks}} E[Y_i] = 2 \times 5 \times \frac{5}{2} = 25$

(b) $Var[X(5)] = \lambda t E[Y_i^2] = 2 \times 5 \times \frac{43}{6} = \frac{215}{3}$

In this example, the support of Y was finite

$\to \mathbb{P}[Y_i = \alpha_j] = p_i \qquad \sum_i p_i = 1$

(it happens the the support of Y_i is finite of countable infinite.

α_j are numbers $j \geq 1$

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

A compound Poisson process arises when events occur following a Poisson process and each event results in a random amount Y_i added to the general sum

Let us say that the i-th event is of type j if $Y_i = \alpha_j$

Let $N_j(t)$ denote the number of events of type j by time t, then, for Proposition 5.2, the random variables $N_j(t)$, $j \geq 1$ are independent Poisson random variables with mean

$$\mathbb{E}[N_j(t)] = \lambda p_j t \quad \text{, respectively}$$

Since for each j, the amount α_j is added to the cumulative sum $N_j(t)$ times by time t, it follows that the cumulative sum can be rewritten as

$$X(t) = \sum_j \alpha_j N_j(t) \qquad (5.26)$$

Example: arrival 1^{st} family of size 1 $\quad j=1$
2^{nd} " " " 4
3^{rd} " " " 4 $\quad \Big\} j=4$

$$X(t) = \sum_{i=1}^{N(t)} Y_i = 1 + 4 + 4 = 9$$

$$X(t) = \sum_j \alpha_j N_j(t) = 1 \times 1 + 0 \times 2 + 0 \times 3 + 2 \times 4 = 9$$

To check equation (5.26), let us compute expected value and variance of $X(t)$

$$E[X(t)] = E\left[\sum_j \alpha_j N_j(t)\right] \overset{!}{=} \sum_j \alpha_j E[N_j(t)] \qquad N_j(t) \sim \text{Poisson}(\lambda p_j t)$$

$$\overset{!}{=} \sum_j \alpha_j \lambda p_j t = \lambda t E[Y_i] \qquad \text{same like (5.26)}$$

$$Var[X(t)] = Var\left[\sum_j \alpha_j N_j(t)\right] \qquad \leftarrow N_j(t) \text{ are independent}$$

$$\overset{!}{=} \sum_j \alpha_j^2 Var[N_j(t)] \qquad Var[\text{Poisson}(\gamma)] = E[\text{Poisson}(\gamma)] = \gamma$$

$$\overset{!}{=} \sum_j \alpha_j^2 \lambda p_j t = \lambda t E[Y_i^2] \qquad \text{same like (5.26)}$$

$$X(t) = \sum_{i=1}^{N(t)} Y_i \underset{\text{finite or countable infinity support of } Y_i}{=} \underline{\sum_j \alpha_j N_j(t)}$$

$$t \to \infty \quad N_j(t) \xrightarrow{t\to\infty} \mathcal{N}$$

The sum of independent normal random variables is a normal r.v.
$X(t)$ can be approximated with a Gaussian distribution for $t \to \infty$

Example 5.28 (continue example 5.26)
With the settings of example 5.26, find the approximate probability that at least 240 individuals migrate to the area within 50 weeks

Sol
from example 5.26: $\lambda = 2 \times$ week, $E[Y_i] = 5/2$, $E[Y_i^2] = 43/6$

$$E[X(50)] = \lambda \cdot t \cdot E[Y_i] = 2 \cdot 50 \cdot 5/2 = 250$$

$$Var[X(50)] = \lambda \cdot t \cdot E[Y_i^2] = 2 \cdot 50 \cdot \frac{43}{6} = \frac{4300}{6}$$

The desired probability is

$$P[X(50) \geq 240] = P\left[\frac{X(50) - 250}{\sqrt{4300/6}} > \frac{240 - 250}{\sqrt{4300/6}}\right]$$

$$\overset{!}{=} 1 - \Phi\left(\frac{240 - 250}{\sqrt{4300/6}}\right)$$

$$\overset{!}{=} 1 - \Phi(-0.3735)$$

$$\overset{!}{=} \Phi(0.3735) \approx 0.6456 \qquad \text{in R pnorm}(0.3735)$$

Another useful result is that $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ are independent compound Poisson process with respective parameters $\lambda_X, F_X, \lambda_Y, F_Y$, then $\{X(t) + Y(t), t \geq 0\}$ is also a compound Poisson process.

This is true because
- the combined process events will occur following a Poisson process with rate $\lambda_X + \lambda_Y$
- each event being from the first compound Poisson process with probability $\frac{\lambda_X}{\lambda_X + \lambda_Y}$

Consequently, the joint compound Poisson process will have parameter $\lambda_X + \lambda_Y$ and distribution F

$$F(z) = \frac{\lambda_X}{\lambda_X + \lambda_Y} F_X(z) + \frac{\lambda_Y}{\lambda_X + \lambda_Y} F_Y(z)$$