Lemma 6.2

(a) $\lim_{h\to 0} \frac{1-P_{ii}(h)}{h} = v_i$

(b) $\lim_{h\to 0} \frac{P_{ij}(h)}{h} = q_{ij}$ $\quad i \neq j$

Lemma 6.3 $\quad \forall t, s \geq 0$

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s)$$

From this lemma

$$P_{ij}(t+h) - P_{ij}(t) = \sum_{k=0}^{\infty} P_{ik}(h) P_{kj}(t) - P_{ij}(t)$$

taking out $k=i$

$$= \sum_{k\neq i} P_{ik}(h) P_{kj}(t) - [1 - P_{ii}(h)] P_{ij}(t)$$

thus

$$\lim_{h\to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \lim_{h\to 0} \Bigg(\sum_{k\neq i} \underbrace{\frac{P_{ik}(h)}{h}}_{h\to 0 \;\to\; q_{ij}} P_{kj}(t) - \underbrace{\frac{[1-P_{ii}(h)]}{h}}_{h\to 0 \;\to\; v_i} P_{ij}(t) \Bigg)$$

by Lemma 6.2

If we can invert limit and summation,

$$P'_{ij}(t) = \sum_{k\neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

Theorem 6.1 (Kolmogorov's backward equations)

$\forall i, j$ (states), $t \geq 0$ (time)

$$P'_{ij}(t) = \sum_{k\neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

Example 6.9

The backward equations for a pure birth process are

$$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) - \lambda_i P_{ij}(t)$$

Example 6.10

The backward equations for a birth and death process are

$$P'_{0j}(t) = \lambda_0 P_{1j}(t) - \lambda_0 P_{0j}(t)$$

$$P'_{ij}(t) = \cancel{(\lambda_i+\mu_i)}\left[\frac{\lambda_i}{\cancel{\lambda_i+\mu_i}} P_{i+1,j}(t) + \frac{\mu_i}{\cancel{\lambda_i+\mu_i}} P_{i-1,j}(t)\right] - (\lambda_i+\mu_i) P_{ij}(t) \quad i>0$$

which can be rewritten as

$$P'_{0j}(t) = \lambda_0 [P_{1j}(t) - P_{0j}(t)] \tag{6.9}$$

$$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i+\mu_i) P_{ij}(t) \quad i>0$$

Example 6.11 (A two-state continuous-time Markov Chain)
We have a machine which works for an exponential time with mean $1/\lambda$ before breaking down. Suppose that it takes an exponential time with mean $1/\mu$ to be repaired. Suppose the machine is working at time 0
?: which is the probability that it will work at time 10?

This is a birth and death process, with two states $\begin{cases} 0, & \text{it is working} \\ 1, & \text{it is broken} \end{cases}$,
and parameters

$$\lambda_0 = \lambda \qquad \mu_0 = 0$$
$$\lambda_1 = 0 \qquad \mu_1 = \mu$$

. Our goal is to compute $P_{00}(10)$

Solving the differential equations of the previous example

$$P'_{00}(t) = \lambda[P_{10}(t) - P_{00}(t)] \quad (6.10) \longrightarrow \mu P'_{00}(t) = \mu\lambda P_{10}(t) - \mu\lambda P_{00}(t) \quad +$$
$$P'_{10}(t) = \mu P_{00}(t) - \mu P_{10}(t) \qquad \lambda P'_{10}(t) = \mu\lambda P_{00}(t) - \mu\lambda P_{10}(t)$$
$$\mu P'_{00}(t) + \lambda P'_{10}(t) = 0$$

By integrating
$$\mu P_{00}(t) + \lambda P_{10}(t) = c$$
where $c = \mu$ ($P_{00}(0) = 1$ and $P_{10}(0) = 0$), therefore
$$\lambda P_{10}(t) = \mu[1 - P_{00}(t)]$$
Substituting the result in (6.10),
$$P'_{00}(t) = \mu - \mu P_{00}(t) - \lambda P_{00}(t) \stackrel{!}{=} \mu - (\mu+\lambda) P_{00}(t)$$

Let $h(t) = P_{00}(t) - \frac{\mu}{\mu+\lambda}$ $\qquad P_{00}(t) = h(t) + \frac{\mu}{\mu+\lambda}$

$$h'(t) = \cancel{\mu} - (\mu+\lambda)\left[h(t) + \cancel{\frac{\mu}{\mu+\lambda}}\right] = -(\mu+\lambda)h(t)$$

Which is
$$\frac{h'(t)}{h(t)} = -(\lambda+\mu)$$
Integrating both sides
$$\log h(t) = -(\lambda+\mu)t + c$$
Taking the exponential
$$h(t) = K e^{-(\mu+\lambda)t}$$
which leads to
$$P_{00}(t) = K e^{-(\mu+\lambda)t} + \frac{\mu}{\mu+\lambda}$$

$t=0$: $1 = K e^0 + \frac{\mu}{\mu+\lambda}$, $\quad K = 1 - \frac{\mu}{\mu+\lambda} = \frac{\lambda}{\mu+\lambda}$

When $t=0$, $P_{00}(0) = 1$, so
$$P_{00}(t) = \frac{\lambda}{\mu+\lambda} e^{-(\mu+\lambda)t} + \frac{\mu}{\mu+\lambda}$$
Computed in $t=10$, we obtain the desired quantity
$$P_{00}(10) = \frac{\lambda}{\mu+\lambda} e^{-10(\mu+\lambda)} + \frac{\mu}{\mu+\lambda}$$

Theorem 6.2 (Kolmogorov's forward equations)
Under suitable regulatory conditions,

$$P_{ij}'(t) = \sum_{k\neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t) \qquad (6.13)$$

Proof
From Lemma 6.3

$$P_{ij}(t+h) - P_{ij}(t) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(h) - P_{ij}(t)$$

take out the term $k=j$

$$= \sum_{k\neq j} P_{ik}(t) P_{kj}(h) - [1 - P_{jj}(h)] P_{ij}(t)$$

and

$$\lim_{h\to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \lim_{h\to 0} \Bigg(\sum_{k\neq j} P_{ik}(t) \underbrace{\frac{P_{kj}(h)}{h}}_{h\to 0 \,\to\, q_{kj}} - \underbrace{\frac{[1 - P_{jj}(h)]}{h}}_{h\to 0 \,\to\, v_j} P_{ij}(t) \Bigg)$$

which, by Lemma 6.2, if we can interchange the limit and the summation (we can do only "under suitable conditions", valid, e.g., for birth and death processes and for all finite-state processes)

$$P_{ij}'(t) = \sum_{k\neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t) \qquad (6.13)$$

Let us solve these equations for a pure birth process.
Equations (6.13) reduces to

$$P_{ij}'(t) = \lambda_{j-1} \underline{P_{i,j-1}}(t) - \lambda_j P_{ij}(t)$$

we cannot go from i to $i-1$

By noting that $P_{ij}(t) = 0$ $\forall j < i$ (because in a pure birth process there are no deaths), we can rewrite

$$P_{ii}'(t) = -\lambda_i P_{ij}(t)$$

$$P_{ij}'(t) = \lambda_{j-1} P_{i,j-1}(t) - \lambda_j P_{ij}(t)$$