6.9 Computing the transition probabilities

$\forall$ states i,j, we let

$$r_{ij} = \begin{cases} q_{ij} & \text{if } i \neq j \\ -v_i & \text{if } i = j \end{cases}$$

The Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

and the Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t)$$

can be rewritten as

$$P'_{ij}(t) = \sum_k r_{ik} P_{kj}(t) \qquad \text{(backward)}$$

$$P'_{ij}(t) = \sum_k r_{kj} P_{ik}(t) \qquad \text{(forward)}$$

respectively.

We can rewrite these equations in matrix form

$$P'(t) = R P(t) \qquad (6.39)$$

$$P'(t) = P(t) R \qquad (6.40)$$

where

$$P'(t) = \begin{array}{c} \\ i\text{th} \end{array} \begin{bmatrix} P'_{kk}(t) & \cdots & \overset{j\text{th}}{\cdots} & \\ \vdots & \ddots & P'_{ij}(t) & \\ & & & \ddots \end{bmatrix}, \quad P(t) = i\text{th} \begin{bmatrix} & \overset{j\text{th}}{P_{ij}(t)} & \end{bmatrix}, \quad R = i\text{th} \begin{bmatrix} & \overset{j\text{th}}{} \\ & r_{ij} \end{bmatrix}$$

It can be shown that the solution of the matrix differential equations (6.39) and (6.40) are

$$P(t) = P(0) e^{Rt}$$

Diff. equations
$f'(t) = f(t)c$
sol $f(t) = f(0)e^{ct}$

Since $P(0) = I$,

$$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix}$$

$$P(t) = e^{Rt}$$

where

$$e^{Rt} = \sum_{n=0}^{\infty} R^n \frac{t^n}{n!} \qquad (6.42)$$

and R^n is R multiplied by itself n times

$$e^{Rt} = \sum_{n=0}^{\infty} R^n \frac{t^n}{n!} \qquad (6.42)$$

Direct use of (6.42) is, in practice, inefficient:
- R contains both positive (q_{ij}) and negative ($-v_i$) elements, which may lead to computer issues;
- we need to compute an infinite sum.

For these reasons, the book suggests two approximations:

Approximation method 1

Instead of e^{Rt}, we use the equivalence

$$e^{Rt} = \lim_{n\to\infty} \left(I + R\frac{t}{n}\right)^n \qquad \text{in general } e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

- advantage: for n large enough, $I + R\frac{t}{n}$ has all nonnegative elements.

Approximation method 2

We can use the identity

$$e^{-Rt} = \lim_{n\to\infty} \left(I - R\frac{t}{n}\right)^n \overset{n \text{ large enough}}{\approx} \left(I - R\frac{t}{n}\right)^n$$

and therefore

$$P(t) = e^{Rt} \approx \left(I - R\frac{t}{n}\right)^{-n} = \left[\left(I - R\frac{t}{n}\right)^{-1}\right]^n$$

also $\left(I - R\frac{t}{n}\right)^{-1}$ has only non-negative elements

Remarks:
- if we use $n = 2^k$, computations are easier (we multiply the resulting matrix by itself k time)
- the two approximations have probabilistic interpretations.

7.1 Renewal process

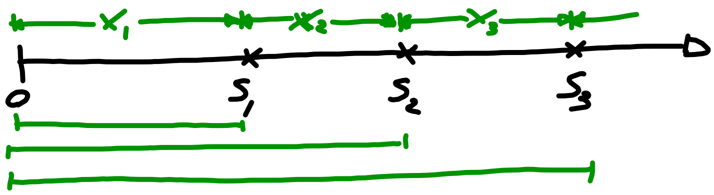
Definition: Let $\{N(t), t \geq 0\}$ be a counting process and let X_n denote the time between the $(n-1)$-th and n-th event of the process, $n \geq 1$. If the sequence of non-negative random variables $\{X_1, X_2, \dots\}$ is independent and identically distributed, then the counting process $\{N(t), t \geq 0\}$ is said to be a renewal process

With respect to a Poisson process, we are relaxing the condition on the distribution of X_n, $n \geq 1$ (they are still iid, but not necessarily exponentially distributed).

For a renewal process having interarrival times $X_1, X_2, \dots$, let

$$S_0 = 0 \quad \wedge \quad S_n = \sum_{i=1}^{n} X_i \quad , n \geq 1$$

S_n denotes the time of the n-th arrival (renewal)



Let F denote the interarrival distribution and $F(0) = \mathbb{P}[X_n = 0] < 1$.
Moreover, let

$$\mu = \mathbb{E}[X_n] \quad , n \geq 1$$

be the mean time between two successive renewals. From the nonnegativity of X_n and $\mathbb{P}[X_n = 0] < 1$, it follows that $\mu > 0$

Can $N(t)$ be infinite for some finite value of t? We show that it cannot

Given S_n, the time of the n-th renewal, $N(t)$ can be written as

$$N(t) = \max\{n : S_n \le t\} \qquad (7.1)$$

By the law of the large numbers, it follows that, with probability 1,

$$\frac{S_n}{n} \longrightarrow \mu \qquad n \longrightarrow \infty$$

Since $\mu > 0$, $S_n = O_p(n)$, $S_n \to \infty$ for $n \to \infty$ with the same speed.

Therefore S_n can be less or equal to t for at most a finite number of values of n, and $N(t)$ is finite in equation (7.1).

However, though $N(t) < \infty$ for each t, it is true with probability 1 that

$$N(\infty) = \lim_{t \to \infty} N(t) = \infty$$

This follows because the only way for $N(\infty)$ to be finite is that one of the arrival time is infinity. So

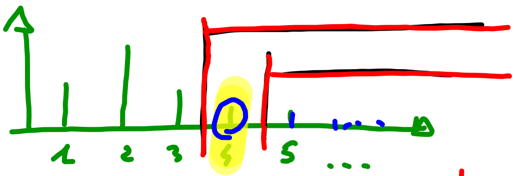
$$\begin{aligned} \mathbb{P}[N(\infty) < \infty] &= \mathbb{P}[X_n = \infty \text{ for some } n] \\ &= \mathbb{P}\Big[\bigcup_{n=1}^{\infty} (X_n = \infty)\Big] \\ &\le \sum_{n=1}^{\infty} \mathbb{P}[X_n = \infty] = 0 \end{aligned}$$

$F(\infty) = 1$

7.2 Distribution $N(t)$

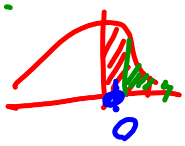
Note that

$N(t) \geq n \iff S_n \leq t$; (7.2)



from that

$$\mathbb{P}[N(t)=n] = \mathbb{P}[N(t)\geq n] - \mathbb{P}[N(t)\geq n+1]$$
$$= \mathbb{P}[S_n \leq t] - \mathbb{P}[S_{n+1} \leq t] \quad (7.3)$$



Since X_i are i.i.d. $(i\geq 1)$, with distribution F, it follows that $S_n = \sum_{i=1}^{n} X_i$ is distributed as F_n, the n-fold convolution of F with itself (section 2.5).

Therefore, from equation (7.3), we obtain

$$\mathbb{P}[N(t)=n] = F_n(t) - F_{n+1}(t)$$

Example 7.1

Suppose that the interarrival distribution is geometric with parameter p, $\mathbb{P}[X_n=i] = p(1-p)^{i-1}$.

Therefore $S_1 = X_1$ is #trials until the first success (geometric)

S_n is the number of trials until the n-th success (negative binomial)

$$\mathbb{P}[S_n=k] = \begin{cases} \binom{k-1}{n-1} p^n (1-p)^{k-n} & \text{if } k\geq n \\ 0 & \text{if } k<n \end{cases}$$

Then, from equation (7.3)

integer part of t

$$\mathbb{P}[N(t)=n] = \sum_{k=n}^{[t]} \binom{k-1}{n-1} p^n (1-p)^{k-n} - \sum_{k=n+1}^{[t]} \binom{k-1}{n} p^{n+1} (1-p)^{k-n-1}$$

or, equivalently, since an event occurs independently at each time 1, 2, ..

$$\mathbb{P}[N(t)=n] = \binom{[t]}{n} p^n (1-p)^{[t]-n}$$

Another expression for $\mathbb{P}[N(t)=n]$ can be obtained by conditioning on S_n,

$$\mathbb{P}[N(t)=n] = \int_0^\infty \mathbb{P}[N(t)=n \mid S_n = y]\, f_{S_n}(y)\, dy$$

$$= \int_0^t \mathbb{P}[N(t)=n \mid S_n=y] f_{S_n}(y)dy + \underbrace{\int_t^\infty \mathbb{P}[N(t)=n \mid S_n=y] f_{S_n}(y)dy}_{=0}$$

if $y>t$, then $N(t)<n$

$$= \int_0^t \mathbb{P}[X_{n+1} > t-y \mid S_n = y] f_{S_n}(y)dy$$

there will be n renewals by time t if the next interarrival time is larger than $t-y$

$$= \int_0^t \mathbb{P}[X_{n+1} > t-y] f_{S_n}(y)dy$$

by independence

$$= \int_0^t [1 - F(t-y)] f_{S_n}(y)dy$$

Example 7.2

If $F(x) = 1 - e^{-\lambda x}$, then S_n is a sum of n iid random variables with exponential distribution with rate λ, i.e. $S_n \sim Gamma(n, \lambda)$

As a consequence

$$\mathbb{P}[N(t)=n] = \int_0^t \underbrace{e^{-\lambda(t-y)}}_{1-F(t-y)} \frac{\overbrace{\lambda e^{-\lambda y} (\lambda y)^{n-1}}^{f_{S_n}(y)}}{(n-1)!} dy$$

density of a gamma distribution

$$= \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \int_0^t y^{n-1} dy = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$\left[\frac{1}{n} y^n\right]_0^t = \frac{1}{n} t^n$$

By using equation (7.2), we can compute $m(t) = E[N(t)]$, the mean value of $N(t)$, as

$$m(t) = E[N(t)] = \sum_{n=1}^{\infty} P[N(t) \geq n] = \sum_{n=1}^{\infty} P[S_n \leq t] = \sum_{n=1}^{\infty} F_n(t)$$

We used the fact that X is nonnegative and integer valued, for which

$$E[X] = \sum_{k=1}^{\infty} \underbrace{k}_{\sum_{n=1}^{k} 1} P[X=k] = \sum_{k=1}^{\infty}\sum_{n=1}^{k} P[X=k] = \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} P[X=k] = \sum_{n=1}^{\infty} P[X \geq n]$$

Definition: $m(t)$ is also called renewal function

It can be shown that:

(i) $m(t)$ uniquely determines the renewal process (i.e., there is an one-to-one correspondence between F and $m(t)$)

(ii) due to (i), the Poisson process is the only renewal process with linear mean-value (renewal) function

(iii) $m(t) < \infty \quad \forall t < \infty$

(iv) result (iii) do not follow from the fact that $N(t)$ is finite with probability 1

An integral equation satisfied by the renewal function can be obtained by conditioning on the time of the first renewal.

Assuming F continuous with density f,

$$m(t) = E[N(t)] = \int_0^{\infty} E[N(t) \mid X_1 = x] f_X(x)\,dx \qquad (7.4)$$

We have

$$E[N(t) \mid X_1 = x] = \underbrace{1}_{\text{the first one occured}} + \underbrace{E[N(t-x)]}_{\text{the renewal process restarts after the first renewal}} \qquad \text{if } x < t$$

$$E[N(t) \mid X_1 = x] = 0 \qquad \text{if } x > t \quad \text{no renewal before } t$$

Hence, from equation (7.4),

$$m(t) = \int_0^{t} [1 + m(t-x)] f(x)\,dx = F(t) + \int_0^{t} m(t-x) f(x)\,dx$$

which is called the renewal equation and can sometimes be used to obtain the renewal function

Example 7.3

One case where the renewal function can be solved to obtain $m(t)$ is that in which the interarrival distribution is uniform, let us say Unif(0,1)

When $t \le 1$, the renewal function becomes

$$m(t) = t + \int_0^t m(t-x) f(x) dx$$

$$= t + \int_0^t m(y) f(y) dy \qquad \text{substituting } y = t - x$$

Differentiating

$$m'(t) = 1 + m(t)$$

Using $h(t) = \underline{1 + m(t)}$ $\qquad h'(t) = m'(t)$

$$h'(t) = h(t)$$

or

$$\frac{h'(t)}{h(t)} = 1$$

or (integrating)

$$\log h(t) = t + c$$

and

$$h(t) = Ke^t$$

or, back to $m(t)$

$$m(t) + 1 = Ke^t$$

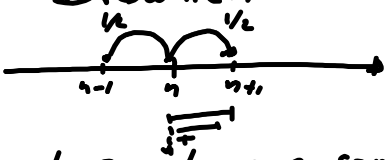
or

$$m(t) = Ke^t - 1$$

Since $m(0) = 0$, $0 = Ke^0 - 1 \Leftrightarrow K = 1$, and so

$$m(t) = e^t - 1 \qquad 0 \le t \le 1$$

10.1 Brownian Motion



Imagine to speed up a random walk by imposing smaller and smaller steps (size Δx) in shorter and shorter interval times (Δt). As for the random walk, the step of size Δx can be on the right or on the left with the same probability $1/2$

If we let $X(t)$ denote the position at time t, then

$$X(t) = \Delta x \left(X_1 + \dots + X_{[t/\Delta t]}\right) \qquad (10.1)$$

$\rightarrow$ integer part of $t/\Delta t$

where

$$X_i = \begin{cases} 1 & i\text{-th step is on the right} \\ -1 & i\text{-th step is on the left} \end{cases} \qquad \forall i$$

with $X_i \perp\!\!\!\perp X_j$, $i \neq j$, and $\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = 1/2$

(independent)

Since $\mathbb{E}[X_i] = 0$ and $Var[X_i] = \mathbb{E}[X_i^2] = 1^2 \times \frac{1}{2} + (-1)^2 \times \frac{1}{2} = 1$, from equation (10.1), we have

$$\mathbb{E}[X(t)] = 0$$

$$Var[X(t)] = (\Delta x)^2 \left[\frac{t}{\Delta t}\right] \qquad \text{by independance}$$

We define Δt and Δx in a clever way, namely $\Delta x = \sigma\sqrt{\Delta t}$, for a positive constant σ. Then, for $\Delta t \to 0$

$$\mathbb{E}[X(t)] = 0$$

$$Var[X(t)] = \sigma^2 \cancel{(\sqrt{\Delta t})^2} \frac{t}{\cancel{\Delta t}} = \sigma^2 t \qquad \Delta t \to 0,\ \left[\frac{t}{\Delta t}\right] \to \infty$$

and we can state, from equation (10.1) and central limit theorem

(i) $X(t) \sim \mathcal{N}(0, \sigma^2 t)$

(ii) $\{X(t), t \geq 0\}$ has independent increments

(iii) $\{X(t), t \geq 0\}$ has stationary increments and $X(t+s) - X(t)$ does not depend on t

Definition 10.1
A stochastic process $\{X(t), t \geq 0\}$ is said to be a Brownian motion process if
(i) $X(0) = 0$
(ii) $\{X(t), t \geq 0\}$ has stationary and independent increments
(iii) $\forall t > 0$, $X(t) \sim \mathcal{N}(0, \sigma^2 t)$

Definition: when $\sigma = 1$, the process is called standard Brownian motion

Note that each Brownian motion can be transformed in a standard Brownian motion by letting $B(t) = \frac{X(t)}{\sigma}$.
Without loss generality, in the following we assume $\sigma = 1$

The interpretation of a Brownian motion as the limit of a random walk suggests that $X(t)$ is a continuous function of t, i.e., with probability 1

$$\lim_{h \to 0} [X(t+h) - X(t)] = 0$$

As $X(t) \sim \mathcal{N}(0, t)$

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$$

By the independent increments assumption,
$X(t_1), X(t_2) - X(t_1), \dots, X(t_n) - X(t_{n-1})$
are independent, and, by the stationary increments assumption,
$X(t_k) - X(t_{k-1}) \sim \mathcal{N}(0, t_k - t_{k-1})$
Therefore, the joint density of $X(t_1), X(t_2), \dots, X(t_n)$ is given by

$$f(x_1, \dots, x_n) = f_{t_1}(x_1) f_{t_2 - t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n - x_{n-1})$$

$$= \frac{\exp\left\{-\frac{1}{2}\left[\frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \dots + \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}\right]\right\}}{(2\pi)^{n/2}[t_1 (t_2 - t_1) \cdots (t_n - t_{n-1})]^{1/2}} \qquad (10.3)$$