Def of periodic Markov chain
A Markov chain that can only return to a state in a multiple of $d \geq 1$ steps is said to be periodic

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Gambler's ruin problem

we saw formula (4.14)

$$P_i = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq q \\ i/N & \text{if } p = q \end{cases}$$

for $N \to \infty$

$$P_i = \begin{cases} 1-(q/p)^i & \text{if } p > q \\ 0 & \text{if } p \leq q \end{cases}$$

Example 4.28

Max and Patty play. Patty wins with probability 0.6

? (a) $\mathbb{P}$[Patty wipes Max | Patty starts with 5 units, Max with 10]

(b) $\mathbb{P}$[" " " | " " " 10 " , " " 20]

Solution

(a) Apply formula (4.14)

$$P = \frac{1-\left(\frac{0.4}{0.6}\right)^5}{1-\left(\frac{0.4}{0.6}\right)^{15}} \approx 0.87$$

(b) Apply formula (4.14)

$i = 10$

$N = 30$

$$P = \frac{1-\left(\frac{0.4}{0.6}\right)^{10}}{1-\left(\frac{0.4}{0.6}\right)^{30}} \approx 0.98$$

Application of the gambler's ruin problem to a drug testing

Suppose to have two drugs, with cure rate P_1 an P_2, and we need to test whether $P_1 > P_2$ or $P_2 > P_1$.
Suppose to give sequentially the drugs to pair of patients, until the number of those who are cured with one drug exceeds the other by a given number

(X_0, Y_0), (X_1, Y_1), ...

$$X_j = \begin{cases} 1 & \text{if the patient receiving drug 1 is cured} \\ 0 & \text{otherwise} \end{cases}$$

$$Y_j = \begin{cases} 1 & \text{if the patient receiving drug 2 is cured} \\ 0 & \text{otherwise} \end{cases}$$

(X_j, Y_j):
- $(1,0)$ → with prob $P_1(1-P_2)$ → goes up of 1
- $(0,1)$ → " " $P_2(1-P_1)$ → goes down of 1
- $(0,0)$ → " " $(1-P_1)(1-P_2)$ } → stays equal
- $(1,1)$ → " " $P_1 P_2$ }

stop when $(X_1 + X_2 + \dots + X_n) - (Y_1 + Y_2 + \dots + Y_n) = \pm M$ — M if $P_1 > P_2$; $-M$ if $P_2 > P_1$

?: Is it a good test? I.e. which is the probability of having an incorrect answer (saying $P_1 > P_2$ when $P_2 > P_1$, or the other way around)

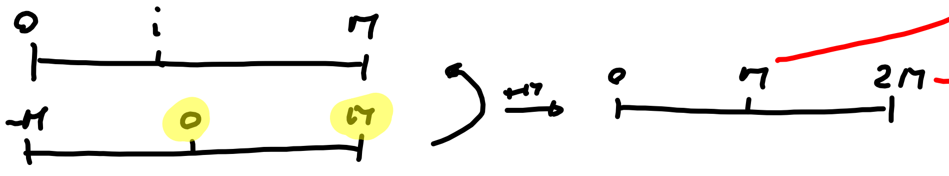
$$p = \mathbb{P}[\underset{\text{controlling}}{\text{number}} \text{ up } 1 \mid \text{up } 1 \vee \text{down } 1] = \frac{P_1(1-P_2)}{P_1(1-P_2) + P_2(1-P_1)}$$

$$q = \mathbb{P}[\underset{\text{controlling}}{\text{number}} \text{ down } 1 \mid \text{up } 1 \vee \text{down } 1] = \frac{P_2(1-P_1)}{P_1(1-P_2) + P_2(1-P_1)}$$

$p + q = 1$

→ the requested probability is the probability that a gambler who wins with probability p reaches $-M$ before M (or M before $-M$ with prob. q)

Using (4.1) $\mathbb{P}[\text{test} \overset{\text{wrongly}}{\text{assert}} \text{ that } P_2 > P_1] = 1 - \dfrac{1-(q/p)^M}{1-(q/p)^{2M}} = 1 - \dfrac{1}{1+(q/p)^M} = \dfrac{1}{1+(p/q)^M}$



For example, if $P_1 = 0.6$ and $P_2 = 0.4$, $\mathbb{P}[\text{wrong decision}] = 0.017$ when $M=5$
$\mathbb{P}[\ \text{"}\ \ \text{"}\ ] = 0.0003$ " $M=10$

4.6 Mean time spent in a transient state

Let $T = \{1, \dots, t\}$ denote the transient states in a finite-state Markov chain

Let

$$P_T = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1t} \\ P_{21} & P_{22} & \cdots & P_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ P_{t1} & P_{t2} & \cdots & P_{tt} \end{bmatrix}$$

- P_T specifies only the transition probabilities from a transient state to a transient state;
- some of P_T's row sums are smaller than 1, because in a finite-state Markov chain not all state can be transient

Example

$$P = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & \cdots & M-1 & M \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ M-1 \\ M \end{array} & \begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ 0.6 & 0 & 0.4 & \cdots & & 0 \\ \vdots & 0.6 & 0 & 0.6 & & \\ & & \ddots & \ddots & \ddots & \\ 0 & & & 0.6 & 0 & 0.4 \\ 0 & \cdots & & & & 1 \end{bmatrix} \end{array}$$

$$P_T = \begin{bmatrix} 0 & 0.4 & 0 & \cdots \\ & & & \\ & & & \end{bmatrix} \leftarrow \sum_j P_{1j} = 0.4 < 1$$

For transient states i and j, let

s_{ij} := expected number of time periods that the Markov chain is in state j given that it starts in state i

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Conditioning on the initial transition

$$s_{ij} = \delta_{ij} + \sum_k P_{ik} s_{kj}$$

$$\overset{!}{=} \delta_{ij} + \sum_{k=1}^{t} P_{ik} s_{kj} \quad (4.18) \qquad s_{kj} = 0 \quad \forall k \notin \{1, \dots, t\}$$

it is impossible to go from a recurrent state to a transient one

$$S = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1t} \\ s_{21} & s_{22} & \cdots & s_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ s_{t1} & s_{t2} & \cdots & s_{tt} \end{bmatrix}$$

Rewrite (4.18) in a matrix form

$$S = I + P_T S \qquad \text{where } I = \overbrace{\begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}}^{t}$$

With simple algebra

$$(I - P_T) S = I \qquad \text{multiplying by } (I - P_T)^{-1}$$

$$S = (I - P_T)^{-1}$$

$\Rightarrow$ the values s_{ij} can be obtain by inverting the matrix $(I - P_T)$

Example 4.30
Consider the gambler's ruin problem with $p = 0.4$ and $N = 7$. Starting from 3 compute:
(a) expected amount of time the gambler has 5 units;
(b) " " " " " " " 2 " .

$$P_T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0.4 & 0 & & & \\ 0.6 & 0 & 0.4 & & 0 & \\ 0 & 0.6 & 0 & 0.4 & & \\ & & 0.6 & 0 & 0.4 & \\ & 0 & & 0.6 & 0 & 0.4 \\ & & & & 0.6 & 0 \end{bmatrix} \end{array}$$

0

$7 = N$

$$S = (I - P_T)^{-1} = \begin{array}{c} \\ \\ 3 \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1.6149 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1.4206 & 2.3677 & 2.9990 & 1.7533 & 0.9228 & 0.3691 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \end{array}$$

(a) $s_{35} = 0.9228$ (b) $s_{32} = 2.3677$

For $i \in T$, $j \in T$, let

f_{ij} := probability that the Markov chain ever makes a transition into state j given that it starts from state i

Conditioning on whether state j is ever entered ($\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]]$)

$$s_{ij} = E[\text{time in } j \mid \text{starts in } i, \text{ ever transits to } j]\, f_{ij} + E[\text{time in } j \mid \text{starts in } i, \text{ never transits to } j]\,(1 - f_{ij})$$

$$= (\delta_{ij} + s_{jj})\, f_{ij} + \delta_{ij}(1 - f_{ij})$$

$$= \delta_{ij} + f_{ij} s_{jj}$$

Solving in f_{ij}

$$f_{ij} = \frac{s_{ij} - \delta_{ij}}{s_{jj}}$$

Example 4.31 (continue example 4.30)

? = which is the probability that the gambler's fortune is ever equal to 1 starting from 3

$$f_{31} = \frac{1.4206}{1.6149} \approx 0.8797$$

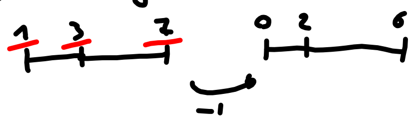
$s_{31} = 1.4206$

$s_{11} = 1.6149$

$i = 3$

$j = 1$

Note that f_{31} is the probability that the gambler's fortune starting at 3 reaches 1 before 7; i.e., the probability that a gambler starting from 2 goes bankrupts before reaching $N=6$



from (4.14) $f_{31} = 1 - \dfrac{1 - \left(\frac{0.6}{0.4}\right)^2}{1 - \left(\frac{0.6}{0.4}\right)^6} \approx 0.8797$

4.7 Branching processes

Consider a population consisting of individuals able to produce offspring of the same kind. Define

$P_j =$ probability that an individual will produce j new offspring during its lifetime, independently of the number produced by the other individuals

$j \geq 0$

Assume $P_j < 1 \quad \forall j \geq 0$

$X_n :=$ size of generation n

$X_0 = \#$ individuals initially present

$X_1 = \#$ " of the first generation

$\vdots$

$\{X_n,\ n \in \{0, \mathbb{N}\}\}$ is a Markov chain

Note:

→ $\emptyset$ is an absorbing state, $P_{00} = 1$ (extinction)

→ if $P_0 > 0$, all other states are transient, since $(P_0)^i = P_{0i}$

→ since any finite set of transient states $\{1, \dots, n\}$ will be visited only finitely often, $P_0 > 0$ then the population either dies or it's size will converge to infinity.

Let

$\mu = \sum_{j=0}^{\infty} j P_j$ denote the average number of offspring of a single individual

$\sigma^2 = \sum_{j=0}^{\infty} (j-\mu)^2 P_j$ denote the variance of the number of offspring of a single individual

Suppose $X_0 = 1$ (i.e., the population initially has 1 individual)

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i$$

where $Z_i :=$ number of offspring of the i-th individual of the $n-1$ generation

Z_i are independent random variables

Conditioning on X_{n-1}

$$\mathbb{E}[X_n] = \mathbb{E}[\mathbb{E}[X_n | X_{n-1}]]$$

$$= \mathbb{E}\Big[\underbrace{\mathbb{E}\Big[\sum_{i=1}^{X_{n-1}} Z_i \,|\, X_{n-1}\Big]}_{\mu+\mu+\dots+\mu \quad X_{n-1} \text{ times}}\Big]$$

$$= \mathbb{E}[X_{n-1}\mu] = \mu\, \mathbb{E}[X_{n-1}]$$

Since $X_0 = 1$

$$\mathbb{E}[X_1] = \mu\, \mathbb{E}[X_0] = \mu$$

$$\mathbb{E}[X_2] = \mu\, \mathbb{E}[X_1] = \mu^2$$

$$\vdots$$

$$\mathbb{E}[X_n] = \mu\, \mathbb{E}[X_{n-1}] = \mu^n$$

About the variance

$$Var[X_n] = E\big[Var[X_n|X_{n-1}]\big] + Var\big[E[X_n|X_{n-1}]\big]$$

Given X_{n-1}, X_n is a sum of independent random variables Z_i:

$$Var[X_n] = E[X_{n-1}\sigma^2] + Var[\mu X_{n-1}]$$
$$\stackrel{!}{=} \sigma^2 E[X_{n-1}] + \mu^2 Var[X_{n-1}]$$
$$\stackrel{!}{=} \sigma^2\mu^{n-1} + \mu^2(\sigma^2\mu^{n-2} + \mu^2 Var[X_{n-2}])$$
$$\stackrel{!}{=} \sigma^2(\mu^{n-1} + \mu^n) + \mu^4 Var[X_{n-2}]$$
$$\stackrel{!}{=} \sigma^2(\mu^{n-1} + \mu^n) + \mu^4(\sigma^2\mu^{n-3} + \mu^2 Var[X_{n-3}])$$
$$\stackrel{!}{=} \sigma^2(\mu^{n-1} + \mu^n + \mu^{n+1}) + \mu^6 Var[X_{n-3}]$$
$$\vdots$$
$$= \sigma^2(\mu^{n-1} + \mu^n + \dots + \mu^{2n-2}) + \mu^{2n} Var[X_0]$$
$$\stackrel{!}{=} \sigma^2(\mu^{n-1} + \mu^n + \dots + \mu^{2n-2})$$

$$(x^n - y^n) = (x^{n-1} + x^{n-2}y + \dots + x y^{n-2} + y^{n-1})$$

Therefore

$$Var[X_n] = \begin{cases} \sigma^2\mu^{n-1}\left(\dfrac{1-\mu^n}{1-\mu}\right) & \text{if } \mu \neq 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases}$$

Let π_0 denote the probability of extinction (starting from 1 individual)
formally $\pi_0 := \lim_{n\to\infty} \mathbb{P}[X_n = 0 \mid X_0 = 1]$

Obviously, if $\mu < 1$ then $\pi_0 = 1$

$\mu^n = E[X_n] = \sum_{j=0}^{\infty} j\,\mathbb{P}[X_n = j] \geq \sum_{j=0}^{\infty} 1\,\mathbb{P}[X_n = j] = \mathbb{P}[X_n \geq 1]$

Since $\mu^n \to 0$ when $\mu < 1$, $\mathbb{P}[X_n \geq 1] \to 0 \Rightarrow \mathbb{P}[X_n = 0] \to 1$

It can be shown that $\pi_0 = 1$ even if $\mu = 1$

If $\mu > 1$, $\pi_0 < 1$; we have

$\pi_0 = \mathbb{P}[\text{population dies out}]$

$\overset{!}{=} \sum_{j=0}^{\infty} \mathbb{P}[\text{population dies out} \mid X_1 = j]\, P_j = \sum_{j=0}^{\infty} \pi_0^j P_j \qquad (4.20)$

It can be shown that π_0 is smallest positive number satisfying (4.20)

Example 4.32

If $P_0 = 1/2$, $P_1 = 1/4$, $P_2 = 1/4$; ?: determine π_0

$\mu = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} = \frac{3}{4} < 1 \implies \pi_0 = 1$

Example 4.33

If $P_0 = 1/4$, $P_1 = 1/4$, $P_2 = 1/2$ from (4.20)

$\mu = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} = \frac{5}{4} > 1 \to \pi_0 = \underbrace{\frac{1}{4}}_{j=0} + \underbrace{\pi_0 \cdot \frac{1}{4}}_{j=1} + \underbrace{\pi_0^2 \cdot \frac{1}{2}}_{j=2}$

$\frac{\pi_0^2}{2} - \overset{1-\frac{1}{4}}{\frac{3}{4}}\pi_0 + \frac{1}{4} = 0 \qquad 2\pi_0^2 - 3\pi_0 + 1 = 0$

$(2\pi_0 - 1)(\pi_0 - 1) = 0 \quad \begin{cases} \pi_0 = 1/2 \\ \pi_0 = 1 \end{cases}$

Example 4.34

Which is the probability in examples 4.32 and 4.33, that a population initially consisting of n individuals die out?

$\text{prob} = (\pi_0)^n \qquad$ ex 4.32 $\pi_0 = 1 \qquad \pi_0^n = 1$

ex 4.33 $\pi_0 = 1/2 \qquad \pi_0^n = 1/2^n$

all families generated from the n individuals die out