Example 5.6 hyperexponential

$$f(t) = \sum_{j=1}^{n} \lambda_j P_j e^{-\lambda_j t} \qquad r(t) = \frac{\sum_{j=1}^{n} \lambda_j P_j e^{-\lambda_j t}}{\sum_{j=1}^{n} P_j e^{-\lambda_j t}} \qquad P_j = P[B=j]$$

Plan for today
- more properties of exponential distribution;
- hypoexponential distribution.

Let $X_1, \ldots, X_n$ be independent and identically distributed (iid) random variables with exponential distribution with rate λ. $S = X_1 + \ldots + X_n$, $S \sim \text{Gamma}(n, \lambda)$

$n=1 \to$ trivial

assume that it is true for $n-1 \to$ show that it is true for n

$$f_{X_1+\ldots+X_{n-1}}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-2}}{(n-2)!} \quad \leftarrow \text{Gamma}(n-1, \lambda)$$

$$f_{X_1+\ldots+X_n}(t) = \int_0^t f_{X_n}(t-s)\, f_{X_1+\ldots+X_{n-1}}(s)\, ds$$

$$= \int_0^t \lambda e^{-(t-s)\lambda} \lambda e^{-\lambda s} \frac{(\lambda s)^{n-2}}{(n-2)!}\, ds$$

$$= \frac{\lambda^n e^{-\lambda t}}{(n-2)!} \int_0^t e^{\cancel{+\lambda s}\ \cancel{-\lambda s}} s^{n-2}\, ds$$

$$= \frac{\lambda^n e^{-\lambda t}}{(n-2)!} \left[\frac{s^{n-1}}{n-1} \right]_0^t = \frac{\lambda^n e^{-\lambda t}}{(n-2)!} \left(\frac{t^{n-1}}{n-1} - 0 \right) = \frac{\lambda e^{-\lambda t} (t \cdot \lambda)^{n-1}}{(n-1)!} \quad \text{qed}$$

Determine the probability that one exponential random variable is smaller than another one. $X_1 \sim Exp(\lambda_1)$, $X_2 \sim Exp(\lambda_2)$ $\quad X_1 \perp\!\!\!\perp X_2$

$$P[X_1 < X_2] = \int_0^{+\infty} P[X_1 < X_2 \mid X_1 = x]\, \lambda_1 e^{-\lambda_1 x}\, dx$$

$$= \int_0^{+\infty} e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x}\, dx = \frac{\lambda_1}{(\lambda_1+\lambda_2)} \underbrace{\int_0^{+\infty} (\lambda_1+\lambda_2) e^{-(\lambda_1+\lambda_2)x}\, dx}_{=1} = \frac{\lambda_1}{\lambda_1+\lambda_2} \qquad (5.5)$$

$f_Y(y)$ $\quad Y \sim Exp(\lambda_1+\lambda_2)$

Suppose that $X_1, \dots, X_n$ are independent exponential random variables with rates $\lambda_1, \dots, \lambda_n$, respectively. The smallest X_i, $i = 1, \dots, n$ is exponential random variable with rate $\sum_{i=1}^n \lambda_i$

$$P[\min_i \{X_i\} > x] = P\left[\bigcap_{i=1}^n (X_i > x)\right] = P[X_i > x,\ \forall i = 1, \dots, n]$$

because independence

$$= \prod_{i=1}^n P[X_i > x]$$

$$= \prod_{i=1}^n e^{-\lambda_i x} = e^{-\sum_{i=1}^n \lambda_i x} \qquad (5.6)$$

Let $X_1, \dots, X_n$ be independent exponential random variables with rates $\lambda_1, \dots, \lambda_n$, respectively. The probability that X_i is the smallest is $\frac{\lambda_i}{\sum_{j=1}^n \lambda_j}$

$$P[X_i = \min_j X_j] = P[X_i < \min_{j \neq i} X_j] \overset{(5.5)}{=} \frac{\lambda_i}{\lambda_i + \underbrace{\sum_{j \neq i} \lambda_j}_{(5.6)}} = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}$$

Proposition. If $X_1, \dots, X_n$ are independent exponential random variables with rates $\lambda_1, \dots, \lambda_n$, respectively, then $\min_i X_i$ $i = 1, \dots, n$ is exponentially distributed with rate $\sum_{i=1}^n \lambda_i$. Moreover, $\min_i X_i$ and the rank order are independent.

Proof of the second statement

$$P[X_{i_1}, \dots, X_{i_n} \mid \min_i X_i > t] = P[X_{i_1} - t < \dots < X_{i_n} - t \mid \min_i X_i > t]$$

$$\overset{\text{memoryless}}{=} P[X_{i_1} < \dots < X_{i_n}]$$

Example 5.8

Suppose that you enter into a post office with two clerks, but both are busy with other customers. You will be served by the first clerk that becomes free. The service times of the two clerks follow exponential distributions with rates λ_1 and λ_2 and they are independent. If we define

$T :=$ amount of time that you spend in the post office

which is $\mathbb{E}[T]$?

Solution

$R_i, i=1,2 :=$ remaining service time for the clerk i

$S :=$ your service time

$$\mathbb{E}[T] = \mathbb{E}[T \mid R_1 < R_2]\,\mathbb{P}[R_1 < R_2] + \mathbb{E}[T \mid R_2 \le R_1]\,\mathbb{P}[R_2 \le R_1]$$

$$\overset{!}{=} \mathbb{E}[T \mid R_1 < R_2]\frac{\lambda_1}{\lambda_1+\lambda_2} + \mathbb{E}[T \mid R_2 \le R_1]\frac{\lambda_2}{\lambda_1+\lambda_2}$$

$$\mathbb{E}[T \mid R_1 < R_2] = \mathbb{E}[S + R_1 \mid R_1 < R_2]$$

$$\overset{!}{=} \mathbb{E}[S \mid R_1 < R_2] + \mathbb{E}[R_1 \mid R_1 < R_2]$$

$$\overset{!}{=} \frac{1}{\lambda_1} + \mathbb{E}[\min_i\{R_1, R_2\}]$$

$\min_i\{R_1, R_2\} \sim Exp(\lambda_1 + \lambda_2)$ (5.6)

$$\overset{!}{=} \frac{1}{\lambda_1} + \frac{1}{\lambda_1+\lambda_2}$$

Similarly

$$\mathbb{E}[T \mid R_2 \le R_1] = \frac{1}{\lambda_2} + \frac{1}{\lambda_1+\lambda_2}$$

Therefore

$$\mathbb{E}[T] = \mathbb{E}[T \mid R_1 < R_2]\,\mathbb{P}[R_1 < R_2] + \mathbb{E}[T \mid R_2 \le R_1]\,\mathbb{P}[R_2 \le R_1]$$

$$\overset{!}{=} \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_1+\lambda_2}\right)\frac{\lambda_1}{\lambda_1+\lambda_2} + \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1+\lambda_2}\right)\frac{\lambda_2}{\lambda_1+\lambda_2}$$

$$\overset{!}{=} \frac{\lambda_1+\lambda_2+\lambda_1}{\cancel{\lambda_1}(\lambda_1+\lambda_2)} \cdot \frac{\cancel{\lambda_1}}{\lambda_1+\lambda_2} + \frac{\lambda_1+\lambda_2+\lambda_2}{\cancel{\lambda_2}(\lambda_1+\lambda_2)} \cdot \frac{\cancel{\lambda_2}}{\lambda_1+\lambda_2}$$

$$\overset{!}{=} \frac{3\lambda_1 + 3\lambda_2}{(\lambda_1+\lambda_2)^2} = \frac{3\cancel{(\lambda_1+\lambda_2)}}{(\lambda_1+\lambda_2)^{\cancel{2}}} = \frac{3}{\lambda_1+\lambda_2}$$

Otherwise

$$E[T] = E[\min\{R_1, R_2\} + S]$$
$$= E[\min\{R_1, R_2\}] + E[S]$$
$$= \frac{1}{\lambda_1+\lambda_2} + E[S]$$

$\min\{R_1, R_2\} \sim Exp(\lambda_1+\lambda_2)$

$R_1 \sim Exp(\lambda_1)$

$R_2 \sim Exp(\lambda_2)$

$$E[S] = E[S \mid R_1 < R_2]\,P[R_1 < R_2] + E[S \mid R_2 \le R_1]\,P[R_2 \le R_1]$$
$$= \frac{1}{\cancel{\lambda_1}} \frac{\cancel{\lambda_1}}{\lambda_1+\lambda_2} + \frac{1}{\cancel{\lambda_2}} \frac{\cancel{\lambda_2}}{\lambda_1+\lambda_2} = \frac{2}{\lambda_1+\lambda_2}$$
$$= \frac{1}{\lambda_1+\lambda_2} + \frac{2}{\lambda_1+\lambda_2} = \frac{3}{\lambda_1+\lambda_2}$$

We saw in Example 5.6 the hyperexponential distribution. Let us now define the hypoexponential random variable

Definition: Let $X_1, \dots, X_n$ be independent exponential random variables with rates $\lambda_1, \dots, \lambda_n$, respectively. Suppose $\lambda_i \neq \lambda_j$ for $i \neq j$, $i,j = 1, \dots, n$. The random variable $\sum_{i=1}^{n} X_i$ is said to be a hypoexponential random variable

Let us find the density function $f(t)$. We start with $n=2$

$$f_{X_1+X_2}(t) = \int_0^t f_{X_1}(s) f_{X_2}(t-s)\, ds$$
$$= \int_0^t \lambda_1 e^{-\lambda_1 s} \lambda_2 e^{-\lambda_2 (t-s)}\, ds$$
$$= \lambda_1 \lambda_2 e^{-\lambda_2 t} \int_0^t e^{-(\lambda_1-\lambda_2)s}\, ds$$
$$= \frac{\lambda_1 \lambda_2}{\lambda_1-\lambda_2} e^{-\lambda_2 t} \int_0^t (\lambda_1-\lambda_2) e^{-(\lambda_1-\lambda_2)s}\, ds$$

$1 - F_S(t) = 1 - e^{-(\lambda_1-\lambda_2)t}$

$$= \frac{\lambda_1 \lambda_2}{\lambda_1-\lambda_2} e^{-\lambda_2 t} \left[1 - e^{-(\lambda_1-\lambda_2)t}\right]$$
$$= \frac{\lambda_1 \lambda_2}{\lambda_1-\lambda_2} e^{-\lambda_2 t} + \frac{\lambda_1 \lambda_2}{\lambda_2-\lambda_1} e^{(-\lambda_1 \cancel{+\lambda_2} \cancel{-\lambda_2})t} = \frac{\lambda_1 \lambda_2}{\lambda_1-\lambda_2} e^{-\lambda_2 t} + \frac{\lambda_1 \lambda_2}{\lambda_2-\lambda_1} e^{-\lambda_1 t}$$
$$= \sum_{i=1}^{n} \lambda_i e^{-\lambda_i t} \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}$$

For $n=2$ we saw that $f(t)$ is in the form

$$\sum_{i=1}^{2} \lambda_i e^{-\lambda_i t} \underbrace{\prod_{j\neq i} \frac{\lambda_j}{\lambda_j-\lambda_i}}_{C_{i,2}}$$

A possible generalization: $f(t) = \sum_{i=1}^{n}\left(\lambda_i e^{-\lambda_i t} C_{i,n}\right)$, where $C_{i,n} = \prod_{j\neq i} \frac{\lambda_j}{\lambda_j-\lambda_i}$

We want to prove this formula by induction. We saw that it is true for $n=2$. Suppose that it is true for n, we want to show that it is true for $n+1$.
Consider a further independent exponential random variable X_{n+1}, $X_{n+1} \sim Exp(\lambda_{n+1})$, where $\lambda_{n+1} \neq \lambda_i$, $i=1,\dots,n$. Without loss of generality, let say $X_1 > X_{n+1}$

$$f_{X_1+\dots+X_{n+1}}(t) = \int_0^t f_{X_1+\dots+X_n}(s)\, f_{X_{n+1}}(t-s)\,ds$$

$$= \int_0^t \sum_{i=1}^{n} \lambda_i e^{-\lambda_i s} C_{i,n}\, \lambda_{n+1} e^{-\lambda_{n+1}(t-s)}\,ds$$

$$= \sum_{i=1}^{n} C_{i,n}\lambda_i\lambda_{n+1} e^{-\lambda_{n+1}t}\int_0^t e^{-(\lambda_i-\lambda_{n+1})s}\,ds$$

$1 - e^{-(\lambda_i-\lambda_{n+1})t}$

$1 - F_S(t)$

$$= \sum_{i=1}^{n} C_{i,n}\lambda_i\lambda_{n+1} e^{-\lambda_{n+1}t}\frac{1}{\lambda_i-\lambda_{n+1}}\int_0^t(\lambda_i-\lambda_{n+1})e^{-(\lambda_i-\lambda_{n+1})s}\,ds$$

$$= \sum_{i=1}^{n}\left\{C_{i,n}\left(\frac{\lambda_i\lambda_{n+1}}{\lambda_i-\lambda_{n+1}} e^{-\lambda_{n+1}t} + \frac{\lambda_i\lambda_{n+1}}{\lambda_i-\lambda_{n+1}} e^{\cancel{-\lambda_{n+1}t}-\lambda_i t+\cancel{\lambda_{n+1}t}}\right)\right\}$$

$$= K_{n+1}\lambda_{n+1}e^{-\lambda_{n+1}t} + \sum_{i=1}^{n} C_{i,n+1}\lambda_i e^{-\lambda_i t} \qquad (5.7)$$

where

$$K_{n+1} = \sum_{i=1}^{n}\frac{C_{i,n}\lambda_i}{\lambda_i-\lambda_{n+1}}$$

which is constant as it does not depend on t. In order to complete the proof we need to show that

$$K_{n+1} = C_{n+1,n+1}$$

Let us consider

$$f_{X_1+\dots+X_{n+1}}(t) = \int f_{X_{\cancel{1}2}+\dots+X_{\cancel{n}n+1}}(s)\, f_{X_{\cancel{n+1}1}}(t-s)\,ds$$

and note that, by (5.7)

$$f_{X_2+\dots+X_{n+1}}(t) = K_1\lambda_1 e^{-\lambda_1 t} + \sum_{i=2}^{n+1} C_{i,n+1}\lambda_i e^{-\lambda_i t} \qquad (5.7b)$$

Equating (5.7) and (5.7b),

$$K_{n+1}\lambda_{n+1}e^{-\lambda_{n+1}t} + \sum_{i=1}^{n} C_{i,n}\lambda_i e^{-\lambda_i t} = K_1\lambda_1 e^{-\lambda_1 t} + \sum_{i=2}^{n+1} C_{i,n+1}\lambda_i e^{-\lambda_i t}$$

$$K_{n+1}\lambda_{n+1}e^{-\lambda_{n+1}t} + C_{1,n}\lambda_1 e^{-\lambda_1 t} = K_1\lambda_1 e^{-\lambda_1 t} + C_{n+1,n+1}\lambda_{n+1}e^{-\lambda_{n+1}t}$$

Multiplying both sides by $e^{\lambda_{n+1}t}$

$$K_{n+1}\lambda_{n+1} + \underbrace{C_{1,n}\lambda_1 e^{-(\lambda_1-\lambda_{n+1})t}}_{t\to\infty \; \to 0} = \underbrace{K_1\lambda_1 e^{-(\lambda_1-\lambda_{n+1})t}}_{t\to\infty \; \to 0} + C_{n+1,n+1}\lambda_{n+1}$$

If we let $t \longrightarrow \infty$

$$K_{n+1}\cancel{\lambda_{n+1}} = C_{n+1,n+1}\cancel{\lambda_{n+1}}$$

which completes our proof.

Therefore, we showed that $S = \sum_{i=1}^{n} X_i$ has density

$$f_S(t) = \sum_{i=1}^{n} C_{i,n}\lambda_i e^{-\lambda_i t} \quad (5.8) \qquad \text{where } C_{i,n} = \prod_{j\neq i}\frac{\lambda_j}{\lambda_j-\lambda_i}$$

If we integrate both sides of (5.8)

$$\int_t^{+\infty} f_S(t)\,dt = \int_t^{+\infty}\sum_{i=1}^{n} C_{i,n}\lambda_i e^{-\lambda_i t}$$

$$\mathbb{P}[S>t] = \sum_{i=1}^{n} C_{i,n}\int_t^{+\infty}\lambda_i e^{-\lambda_i t}\,dt = \sum_{i=1}^{n} C_{i,n}e^{-\lambda_i t} = 1-F_S(t)$$

Combining (5.8) with the last equation

$$r_S(t) = \frac{f_S(t)}{1-F_S(t)} = \frac{\sum_{i=1}^{n} C_{i,n}\lambda_i e^{-\lambda_i t}}{\sum_{i=1}^{n} C_{i,n}e^{-\lambda_i t}}$$

if $i\neq j$ $\quad e^{-(\lambda_i-\lambda_j)t} \xrightarrow{\;o\;} 0$

if $i=j$ $\quad e^{-0t} \longrightarrow 1$

Let $\lambda_j = \min_j\{\lambda_1,\dots,\lambda_n\}$. Multiplying numerator and denominator by $e^{+\lambda_j t}$, we get

$$\lim_{t\to\infty} r_S(t) = \lambda_j$$

<u>Remark</u>

Although

$$1 = \int_0^{+\infty} f_S(t)\,dt = \sum_{i=1}^{n} C_{i,n} = \sum_{i=1}^{n}\prod_{i\neq j}\frac{\lambda_j}{\lambda_j-\lambda_i}$$

Note that $C_{i,n}$, $i=1,\dots,n$ are not probabilities, as some of them will be negative.

Hypoexponential and hyperexponential are very different random variables.