8 March, 2017

STK2130

Mandatory assignment

Submission deadline

Thursday 23rd March 2017, 14:30 in the mandatory activity hand-in box, situated on the 7th floor of Niels Henrik Abels hus.

Instructions

You may write your answers either by hand or on a computer (for instance with LaTeX). All submissions must include the following official front page:

http://www.uio.no/english/studies/admin/compulsory-activities/mn-math-obligforside-eng.pdf

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you understand the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code. In order to print the code from one of the Linux machines belonging to the university, move to the folder containing your program and type

```
lpr -P pullprint_manufacturer filename
```

where `filename` is the file you wish to print and `pullprint_manufacturer` is the name of the manufacturer of the printer you wish print from. Common choices are `pullprint_Ricoh` and `pullprint_HP`.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (7th floor of Niels Henrik Abels hus, e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

The assignment consists of 3 Problems. Make sure you have the complete assignment.

Problem 1. A Markov chain $X_0, X_1, X_2, \ldots$ on the states $\{0, 1, 2, 3, 4\}$ is defined by the transition matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) The chain has three classes, $\mathcal{C}_0 = \{0, 1\}, \mathcal{C}_1 = \{2, 3\}, \mathcal{C}_2 = \{4\}$ where $\mathcal{C}_0$ is transient, $\mathcal{C}_1$ is closed and $\mathcal{C}_2$ is absorbing (and closed). Explain why this is so.

Which of the classes are recurrent?

(b) Let T be the time until the chain enters one of the closed classes and define $\mu_i = \mathrm{E}[T|X_0 = i]$ for $i \in \mathcal{C}_0$.

Explain why the μ_i satisfy the following two equations

$$(1) \qquad \mu_0 = (\mu_0 + 1)\tfrac{1}{5} + (\mu_1 + 1)\tfrac{1}{5} + \tfrac{3}{5}$$

$$(2) \qquad \mu_1 = (\mu_0 + 1)\tfrac{2}{5} + (\mu_1 + 1)\tfrac{1}{5} + \tfrac{2}{5}$$

Solve the equations (1) and (2) to obtain μ_0 and μ_1.

(c) Let q_i be the probability that the chain ends up in state 4 conditional on $X_0 = i$. Find and explain equations for obtaining q_0 and q_1. Solve the equations.

d) Let s_{ij} denote the expected number of visits to states $j = 0$ and $j = 1$ conditional on $X_0 = i$. Find and solve equations for determining the s_{ij}.

Problem 2. Let a Markov chain $X_0, X_1, X_2, \ldots$ on the states $\{0, 1, 2, 3, 4\}$ be described by the transition matrix

$$\mathbf{P} = \begin{bmatrix} q & p & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ q & 0 & 0 & p & 0 \\ q & 0 & 0 & 0 & p \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $0 < p < 1$ and $q = 1 - p$.

(a) Explain why this Markov chain has a stationary distribution with limit probabilities $\pi_j = \lim_{n\to\infty} \mathrm{P}(X_n = j|X_0 = i)$ determined by equations

$$(1) \qquad \pi_j = p\pi_{j-1} \quad \text{for } j = 1, 2, 3, 4$$

$$(2) \qquad \textstyle\sum_{j=0}^{4} \pi_j = 1$$

(b) Show that the solution to the equations in (a) are given by

$$\pi_j = \frac{1-p}{1-p^5} p^j \quad \text{for } j = 0, 1, 2, 3, 4$$

(c) Let V_n^j be the indicator that $X_n = j$ (i.e. $V_n^j = 1$ if $X_n = j$ and $V_n^j = 0$ otherwise).

What is the limit of $\bar{V}_n^j = \frac{1}{n} \sum_{i=1}^{n} V_i^j$? when $n \to \infty$?

(d) For some function $h()$ state the limit of $\frac{1}{n} \sum_{i=1}^{n} h(X_i)$ when $n \to \infty$?

Evaluate the limits numerically for (i) $h(x) = x$ and (ii) $h(x) = \exp(x)$ when $p = 0.3$.

(e) Using again $p = 0.3$, calculate $\mathbf{P}^n$ for $n = 2, 4$ and 8, where $\mathbf{P}^n$ is the product of the transition matrix $\mathbf{P}$ by itself n times.

What does the elements P_{ij}^n of $\mathbf{P}^n$ signify?

(f) Compare the $\mathbf{P}^n$ computed in question (e) with the limiting distribution that you derived in question (b).

Problem 3. Consider a random variable X with finite support $\mathcal{X} = \{1, 2, 3, 4\}$ and probability density function

$$\pi = (P[X = 1], P[X = 2], P[X = 3], P[X = 4]) = (0.1, 0.2, 0.3, 0.4)$$

(a) Starting from the uniform proposal distribution $q_{ij} = P[X_{n+1} = j | X_n = i] = 0.25$, compute the transition probability matrix of a time-reversible Markov chain with stationary probabilities π.

(b) Using a statistical software, generate 1000 elements from the time-reversible Markov chain obtained in the previous question and use them to compute $E[\log(X)]$. Report your code, an histogram of the generated elements and the estimate of $E[\log(X)]$.