



Def: Let $f_i = P(\exists n > 0 \text{ s.t. } X_n = i \mid X_0 = i)$, then:

A state i is recurrent if $f_i = 1$.

A state i is transient if $f_i < 1$.

Prop: State i is recurrent if $\sum_{n=1}^{\infty} P_{ii}^n = \infty$

State i is transient if $\sum_{n=1}^{\infty} P_{ii}^n < \infty$

Ex 4.14

* $P_1 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ $\xrightarrow{\text{Eigenvalue decomp}}$ $P_1 = A D A^{-1}$, where:

$D = \begin{bmatrix} -\frac{1}{2} & & \\ & -\frac{1}{2} & \\ & & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

Then $P_1^n = A D^n A^{-1} = A \begin{bmatrix} \frac{1}{2^n} & 0 & 0 \\ 0 & \frac{1}{2^n} & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1}$

$$= \frac{1}{3} \begin{bmatrix} \frac{2^n - 2}{2^n} & \frac{2^n + 2}{2^n} & \frac{2^n + 2}{2^n} \\ \frac{2^n + 2}{2^n} & \frac{2^n - 2}{2^n} & \frac{2^n + 2}{2^n} \\ \frac{2^n + 2}{2^n} & \frac{2^n + 2}{2^n} & \frac{2^n + 2}{2^n} \end{bmatrix}$$

But $\sum_{n=1}^{\infty} \frac{2^n - 2}{2^n} = \sum_{n=1}^{\infty} \left(1 - \frac{2}{2^n}\right) = \infty$. Also $\sum_{n=1}^{\infty} \frac{2^n + 2}{2^n} = \infty$.

Hence all states are recurrent since all states communicate.

$$P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Let's find the classes first this time.

$$\begin{array}{l} \bullet 1 \rightsquigarrow 4 \\ \bullet 2 \rightsquigarrow 4 \\ \bullet 3 \rightsquigarrow 1 ; 3 \rightsquigarrow 2 \\ \bullet 4 \rightsquigarrow 3 \end{array} \left. \vphantom{\begin{array}{l} \bullet 1 \rightsquigarrow 4 \\ \bullet 2 \rightsquigarrow 4 \\ \bullet 3 \rightsquigarrow 1 ; 3 \rightsquigarrow 2 \\ \bullet 4 \rightsquigarrow 3 \end{array}} \right\} \Rightarrow \begin{cases} 4 \rightsquigarrow 3 \rightsquigarrow 2 \Rightarrow 4 \rightsquigarrow 2 \\ 4 \rightsquigarrow 3 \rightsquigarrow 1 \Rightarrow 4 \rightsquigarrow 1 \\ 3 \rightsquigarrow 1 \rightsquigarrow 4 \Rightarrow 3 \rightsquigarrow 4 \\ 1 \rightsquigarrow 2 \rightsquigarrow 3 \rightsquigarrow 4 \end{cases}$$

Since transience/recurrence is a class property, we now need only study one state. Taking $i=2$ for example we find that all states are recurrent.

$$P_3 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{array}{l} \bullet 4 \rightsquigarrow 5 \\ \bullet 1 \rightsquigarrow 3 \\ \bullet 2 \rightsquigarrow 1 ; 2 \rightsquigarrow 3 \\ \bullet 3 \rightsquigarrow 1 \end{array} \left. \vphantom{\begin{array}{l} \bullet 4 \rightsquigarrow 5 \\ \bullet 1 \rightsquigarrow 3 \\ \bullet 2 \rightsquigarrow 1 ; 2 \rightsquigarrow 3 \\ \bullet 3 \rightsquigarrow 1 \end{array}} \right\} \Rightarrow \begin{cases} \bullet 4 \rightsquigarrow 5 \\ \bullet 1 \rightsquigarrow 3 \\ \bullet 2 \end{cases}$$

Thus: $\{1, 3\}$, $\{4, 5\}$, $\{2\}$.

From P_3 we can see that $\{2\}$ is transient while the other two classes are recurrent.

Ex 4.20 $\{X_i\}_{i \in \mathbb{N}} \subset \{0, \dots, M\}$.

X_i is irreducible & aperiodic \Rightarrow Ergodic $\Rightarrow \exists!$ stationary distribution

$$\Rightarrow \exists! \pi = (\pi_0, \dots, \pi_M)^t \text{ s.t. } \pi = \pi P \text{ i.e. } \pi_j = \sum_{i=0}^M \pi_i P_{ij}$$

But $\sum_{i=0}^M \frac{1}{M+1} P_{ij} = \frac{1}{M+1} \underbrace{\sum_{i=0}^M P_{ij}}_{\text{Doubly stochastic}} = \frac{1}{M+1}$

Hence $\pi = \left(\frac{1}{M+1}, \dots, \frac{1}{M+1} \right)^t$ are the long-run proportions.

4.23

X_i is a MC with values in $S = \{G, B\}$ and index set $\{0, 1, \dots\}$ representing the years ahead. The transition matrix:

$$\begin{array}{c} \begin{array}{cc} & \begin{array}{c} G & B \end{array} \\ \begin{array}{c} G \\ B \end{array} & \begin{bmatrix} 0.5 & 0.5 \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{array}$$

Moreover: * # of storms in $N_j \sim P(\lambda)$
 * λ depends on $X_i = G$ or $X_i = B$
 s.t. $\begin{cases} \lambda = 1, & X_i = G \\ \lambda = 3, & X_i = B \end{cases}$

* $X_0 = G$

a) Let N_i denote # of storms in year i . Then:

$$E[N_1 + N_2] = E[E[N_1 | X_1]] + E[E[N_2 | X_2]]$$

$$E[E[N_1 | X_1]] = \sum_{j \in S} E[N_1 | X_1 = j] \cdot P(X_1 = j)$$

$$= \sum_{j \in S} E[N_1 | X_1 = j] \cdot \left(\sum_{i \in S} P(X_1 = j | X_0 = i) P(X_0 = i) \right)$$

$$= \sum_i \sum_j E[N_1 | X_1 = j] P(X_1 = j | X_0 = i) P(X_0 = i)$$

$$= \sum_j E[N_1 | X_1 = j] P(X_1 = j | X_0 = G) \underbrace{P(X_0 = G)}_{=1}$$

$$= E[N_1 | X_1 = G] \cdot P_{GG} + E[N_1 | X_1 = B] P_{GB}$$

$$= 1 \times \frac{1}{2} + 3 \times \frac{1}{2} = 4 \times \frac{1}{2} = 2$$

$$E[E[N_2 | X_2]] = E[N_2 | X_2 = G] \cdot P_{GG}^2 + E[N_2 | X_2 = B] P_{GB}^2$$

$$= 1 \cdot P_{GG}^2 + 3 \cdot P_{GB}^2 = \frac{13}{6}, \text{ since}$$

$$P^2 = \begin{bmatrix} 0.5 & 0.5 \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}^2 = \begin{bmatrix} \frac{5}{12} & \frac{7}{12} \\ \frac{7}{18} & \frac{11}{18} \end{bmatrix}$$

$$\text{Hence: } E[N_1 + N_2] = 2 + \frac{13}{6} = \frac{25}{6}$$

$$\begin{aligned}
 b) \quad P(N_3=0) &= \sum_{j \in S} P(N_3=0 | X_3=j) P(X_3=j) \\
 &= \sum_{j \in S} P(N_3=0 | X_3=j) P(X_3=j | X_0=G) P(X_0=G) \\
 &= P(N_3=0 | X_3=G) P(X_3=G | X_0=G) + \\
 &\quad P(N_3=0 | X_3=B) P(X_3=B | X_0=G) \\
 &= P(N_3=0 | X_3=G) P_{GG}^3 + P(N_3=0 | X_3=B) P_{GB}^3
 \end{aligned}$$

$$\text{Since } \begin{cases} P(N_3=0 | X_3=G) = P(N_3=0 | \lambda=1) = \frac{1^0 e^{-1}}{0!} \\ P(N_3=0 | X_3=B) = \frac{3^0 e^{-3}}{0!} \end{cases}$$

$$\text{we get: } P(N_3=0) = e^{-1} P_{GG}^3 + e^{-3} P_{GB}^3 = 0.17$$

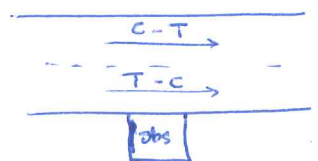
$$\begin{aligned}
 c) \quad "E[N_\infty]" &= E[N_\infty | X_\infty=G] \cdot P(X_\infty=G) + E[N_\infty | X_\infty=B] \cdot P(X_\infty=B) \\
 &= E[N_\infty | X_\infty=G] \cdot \pi_G + E[N_\infty | X_\infty=B] \cdot \pi_B
 \end{aligned}$$

$$\text{Let's find } \pi = (\pi_G, \pi_B): \quad \pi = \pi \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \quad \& \quad \pi_G + \pi_B = 1$$

$$\Rightarrow \pi_G \cdot \frac{1}{2} + \pi_B \cdot \frac{1}{3} = \pi_G \Rightarrow \pi_B = \frac{3}{2} \pi_G \Rightarrow \begin{cases} \pi_G = \frac{2}{5} \\ \pi_B = \frac{3}{5} \end{cases}$$

$$\text{Thus } "E[N_\infty]" = 1 \cdot \frac{2}{5} + 3 \cdot \frac{3}{5} = \frac{11}{5}$$

4.30 | We can construct a Markov chain that represents the state of what an "observer is observing".



Then we have the following transition matrix:

$$P = \begin{matrix} & \begin{matrix} C \\ T \end{matrix} \\ \begin{matrix} C \\ T \end{matrix} & \begin{bmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} \end{matrix}$$

To solve our problem, we find $\pi = \pi P$, i.e.:

$$\begin{cases} \pi_C + \pi_T = 1 \\ \pi_T = \pi_C P_{CT} + \pi_T P_{TT} \end{cases} \Rightarrow \begin{cases} \pi_C + \pi_T = 1 \\ \pi_T = \pi_C \frac{1}{5} + \pi_T \frac{1}{4} \end{cases}$$

$$\Rightarrow \begin{cases} \pi_C + \pi_T = 1 \\ \pi_T = \pi_C \frac{1/5}{1/4} = \pi_C \frac{4}{5} \end{cases} \Rightarrow \begin{cases} \pi_C = \frac{15}{19} \\ \pi_T = \frac{4}{19} \end{cases}$$