

4.35

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix} \end{matrix}$$

Limiting probabilities are solutions to  $\pi P = \pi$ .

4.52

$$P = \begin{matrix} & \begin{matrix} A & B \end{matrix} \\ \begin{matrix} A \\ B \end{matrix} & \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} \end{matrix}$$

Let  $(\text{profit})_n$  denote the profit the driver makes at step (ride) number  $n$ .

Then the quantity of interest is  $E[(\text{profit})_n]$  when  $n$  goes to  $\infty$ .

$$E[(\text{profit})_n] = E[E[(\text{profit})_n | (\text{ride})_n]] \quad \left( (\text{ride})_n := (X_n, X_{n-1}) \right)$$

$$= E[E[(\text{profit})_n | (X_n, X_{n-1})]]$$

↑ random variables

$$= \sum_{i,j} E[(\text{profit})_n | (X_n, X_{n-1}) = (i, j)] \cdot P((X_n, X_{n-1}) = (i, j))$$

$$= \sum_{i,j} E[(\text{profit})_n | (X_n, X_{n-1}) = (i, j)] \cdot P(X_n = j | X_{n-1} = i) P(X_{n-1} = i)$$

$$= E[(\text{profit})_n | (X_n, X_{n-1}) = (A, A)] \cdot P_{AA} \cdot P(X_{n-1} = A)$$

$$+ E[(\text{profit})_n | (X_n, X_{n-1}) = (A, B)] \cdot P_{BA} \cdot P(X_{n-1} = B)$$

$$+ E[(\text{profit})_n | (X_n, X_{n-1}) = (B, A)] \cdot P_{AB} \cdot P(X_{n-1} = A)$$

$$+ E[(\text{profit})_n | (X_n, X_{n-1}) = (B, B)] \cdot P_{BB} \cdot P(X_{n-1} = B)$$

$$= 6 \cdot P_{AA} \cdot P(X_{n-1} = A) + 12 P_{BA} \cdot P(X_{n-1} = B) + 12 P_{AB} \cdot P(X_{n-1} = A) + 8 \cdot P_{BB} \cdot P(X_{n-1} = B)$$

Thus:  $E[\text{profit}] := \lim_{n \rightarrow \infty} E[(\text{profit})_n]$

$$= 6 \cdot P_{AA} \cdot P(X_{\infty} = A) + 12 \cdot P_{BA} \cdot P(X_{\infty} = B) + 12 P_{AB} \cdot P(X_{\infty} = A)$$

$$+ 8 P_{BB} \cdot P(X_{\infty} = B)$$

$$= 6 P_{AA} \pi_A + 12 P_{BA} \pi_B + 12 P_{AB} \pi_A + 8 P_{BB} \pi_B$$

where  $\pi$  are the long-time (stationary) distributions (sol of  $\pi P = \pi$ ).

Ex 4.60

a) Let  $f_i^{(n)} = P(X_1 \neq 3,4; \dots; X_{n-1} \neq 3,4; X_n = 3 \mid X_0 = i)$

then we are seeking  $f_i := \sum_{n=1}^{\infty} f_i^{(n)}$

Now

$$\begin{aligned}
 f_i^{(n)} &= \sum_{j \neq 3,4} P(X_2 \neq 3,4; \dots; X_n = 3 \mid X_1 = j, X_0 = i) P_{ij} \\
 &= \sum_{j \neq 3,4} P(X_2 \neq 3,4; \dots; X_n = 3 \mid X_1 = j) P_{ij} \quad (\text{Markov ppty}) \\
 &= \sum_{j \neq 3,4} P(X_1 \neq 3,4; \dots; X_{n-1} = 3 \mid X_0 = j) P_{ij} \quad (\text{Time homogen}) \\
 &= \sum_{j \neq 3,4} P_{ij} f_j^{(n-1)}
 \end{aligned}$$

Now:

$$\begin{aligned}
 f_i &= f_i^{(1)} + \sum_{n \geq 2} f_i^{(n)} \\
 &= f_i^{(1)} + \sum_{n \geq 2} \sum_{j \neq 3,4} P_{ij} f_j^{(n-1)} \\
 &= f_i^{(1)} + \sum_{j \neq 3,4} P_{ij} \sum_{n \geq 2} f_j^{(n-1)} \\
 &= f_i^{(1)} + \sum_{j \neq 3,4} P_{ij} \underbrace{\sum_{n \geq 1} f_j^{(n)}}_{f_j} \\
 &= f_i^{(1)} + P_{i1} f_1 + P_{i2} f_2.
 \end{aligned}$$

Thus:  $f_i = f_i^{(1)} + P_{i1} f_1 + P_{i2} f_2 \quad i = 1, 2.$

i.e.

$$\begin{cases}
 f_1 = f_1^{(1)} + P_{11} f_1 + P_{12} f_2 \\
 f_2 = f_2^{(1)} + P_{21} f_1 + P_{22} f_2
 \end{cases}$$

with  $f_1^{(1)} = P_{13}$  &  $f_2^{(1)} = P_{23}$

hence

$$\begin{cases}
 f_1 = P_{13} + P_{11} f_1 + P_{12} f_2 \\
 f_2 = P_{23} + P_{21} f_1 + P_{22} f_2
 \end{cases}
 \implies
 \begin{cases}
 f_1 = \frac{11}{21} \\
 f_2 = \frac{8}{21}
 \end{cases}$$

Note that  $f_3 = 1$  and  $f_4 = 0$  and hence we

can write our system as follows:

$$\begin{cases} f_1 = P_{11}f_1 + P_{12}f_2 + P_{13}f_3 + P_{14}f_4 \\ f_2 = P_{21}f_1 + P_{22}f_2 + P_{23}f_3 + P_{24}f_4 \\ f_3 = 0f_1 + 0f_2 + 1f_3 + 0f_4 \\ f_4 = 0f_1 + 0f_2 + 0f_3 + 1f_4 \end{cases}$$

Thus  $f = [f_1, \dots, f_4]^t$  is the solution to  $f = \tilde{P}f$  where

$$\tilde{P} = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

With the condition:

$$f = [f_1, f_2, \underset{\substack{\uparrow \\ \text{reach} \\ 3 \\ \text{first}}}{1}, 0]^t$$

b) Let  $T := \inf_{n \geq 0} \{n \geq 0 : X_n = 3 \text{ or } 4\}$  = first time to hit 3 or 4.

Then:

$$m_i := E_i[T] = E[T | X_0 = i]$$

and

$$m_i = E_i[E[T | X_1]] = E[T_2 | X_1 = 1]P_{i1} + E[T | X_1 = 2]P_{i2} + E[T | X_1 = 3]P_{i3} + E[T | X_1 = 4]P_{i4}$$

but  $E[T | X_1 = i] = 1 + m_i$

Thus

$$m_i = 1 + m_1 P_{i1} + m_2 P_{i2} + \cancel{m_3 P_{i3}} + \cancel{m_4 P_{i4}}$$

$$\Rightarrow \begin{cases} m_1 = 1 + P_{11}m_1 + P_{12}m_2 \\ m_2 = 1 + P_{21}m_1 + P_{22}m_2 \end{cases}$$

$$\Rightarrow \dots \Rightarrow \boxed{m_1 = \frac{55}{21}}$$