4.69 M balls at any point in time distributed among m urns

$$\Leftrightarrow \left(i \in S \Leftrightarrow i = (n_1^i, \dots, n_m^i) \text{ s.t } \sum_{k=1}^{m} n_k^i = M \right)$$

Thus we can make the following guess for π, namely:

$$\pi_i = \frac{M!}{(n_1^i)! \dots (n_m^i)!} \left[\frac{1}{m}\right]^{n_1^i} \times \dots \times \left[\frac{1}{m}\right]^{n_m^i}$$

prob of choosing urn..

$$\Rightarrow \quad \pi_i = \frac{M!}{(n_1^i)! \dots (n_m^i)!} \cdot \left[\frac{1}{m}\right]^M$$

Let's check reversibility i.e $\pi_i P_{ij} \overset{?}{=} \pi_j P_{ji}$

Then it is easy to see that $P_{ij} \neq 0$ iff $\exists\, r, s \in \{1, \dots, m\}$ s.t

$$i = (n_1, \dots, n_r, \dots, n_s, \dots, n_m) \quad \& \quad j = (n_1, \dots, n_r - 1, \dots, n_s + 1, \dots, n_m).$$

Thus:

$$P_{ij} = \frac{n_r}{M} \cdot \frac{1}{m-1} \quad \& \quad P_{ji} = \frac{n_s + 1}{M} \cdot \frac{1}{m-1}$$

and

$$\pi_i P_{ij} = \pi_j P_{ji} \Leftrightarrow \frac{n_r}{M} \cdot \cancel{\frac{1}{m-1}} \cdot \frac{M!}{(n_1)! \dots (n_m)!} \cancel{\left[\frac{1}{m}\right]^M} = \frac{n_s+1}{\cancel{(m-1)}M} \cancel{\left[\frac{1}{m}\right]^M} \frac{M!}{n_1! \dots (n_r - 1)! \dots (n_s+1)! \dots n_m!}$$

$$\Leftrightarrow \frac{n_r}{n_r!\, n_s!} = \frac{n_s + 1}{(n_s+1)!\,(n_r - 1)!} \Leftrightarrow \frac{1}{n_r!\,(n_s+1)!} = \frac{1}{(n_s+1)!\, n_r!} \Leftrightarrow 1 = 1.$$

Thus our chain is reversible.

4.73

$$\begin{array}{c|c} & \begin{array}{ccc} \text{Sunny} & \text{cloudy} & \text{rainy} \\ 0 & 1 & 2 \end{array} \\ \hline \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} =: P \end{array} \longrightarrow \pi P = \pi \text{ has the following sol: } \pi = \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$$

Now it is easy to check that $\forall\, i, j \in \{0, 1, 2\}$ $\pi_i P_{ij} = \pi_j P_{ji}$

For example: $\pi_0 P_{02} = \frac{1}{5} \cdot \frac{1}{2}$ and $\pi_2 P_{20} = \frac{2}{5} \cdot \frac{1}{4}$

thus $\pi_0 P_{02} = \pi_2 P_{20}$.

Ex 3, Exam 2010

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & (1-p) & p & 0 \\ 1 & (1-p) & 0 & p \\ 2 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array} \qquad p \in (0,1)$$

a) There is only one class (communicating), hence all states are recurrent.
The period for state 0 is 1 and hence also for the whole chain.

b) $P_{00}^{(2)} = \sum_i P_{0i} \cdot P_{i0} = P_{00} \cdot P_{00} + P_{01} P_{10} + P_{02} P_{20} = (1-p)^2 + p(1-p) + 0$

$$\Rightarrow \boxed{P_{00}^{(2)} = 1 - p}$$

c) $$P(X_{88} = j \mid X_{87} = 0, X_{89} = 0) = \frac{P(X_{89} = 0, X_{88} = j, X_{87} = 0)}{P(X_{89} = 0, X_{87} = 0)}$$

$$= \frac{P(X_{89} = 0 \mid X_{88} = j, X_{87} = 0) \cdot P(X_{88} = j \mid X_{87} = 0) P(X_{87} = 0)}{P(X_{89} = 0 \mid X_{87} = 0) P(X_{87} = 0)}$$

$$= \frac{P_{j0} \cdot P_{0j}}{P_{00}^{(2)}}$$

Hence we find: $$\begin{cases} P(X_{88} = 0 \mid X_{87} = 0, X_{89} = 0) = \frac{P_{00} P_{00}}{P_{00}^{(2)}} = (1-p). \\ P(X_{88} = 1 \mid X_{87} = 0, X_{89} = 0) = p. \\ P(X_{88} = 2 \mid X_{87} = 0, X_{89} = 0) = 0 \end{cases}$$

d) $$\pi = \pi P \Rightarrow \begin{cases} \pi_0 = \pi_0 (1-p) + \pi_1 (1-p) + \pi_2 \cdot \frac{1}{3} \\ \pi_1 = \pi_0 p + \pi_1 \cdot 0 + \pi_2 \cdot \frac{1}{3} \\ \pi_2 = \pi_0 \cdot 0 + \pi_1 \cdot p + \pi_2 \cdot \frac{1}{3} \end{cases} + \boxed{\sum_{i=0}^{2} \pi_i = 1}$$

e) $$E[T_0 \mid X_0 = 0] = \frac{1}{\pi(0)}$$

f) $u_i = E[W \mid X_0 = i]$ where $W = \min\{n \geq 0 : X_n = 2\}$. Thus $u_2 = 0$

$$u_1 = E[W \mid X_0 = 1] = P_{10} \cdot (1 + u_0) + P_{11} \cdot (1 + u_1) + P_{12} \cdot (1 + u_2)$$

$$u_0 = E[W \mid X_0 = 0] = P_{00} (1 + u_0) + P_{01} (1 + u_1) + P_{02} (1 + u_2)$$

$$\Rightarrow \begin{cases} u_1 = p + (1-p)(1 + u_0) \\ u_0 = (1-p)(1 + u_0) + p(1 + u_1) \end{cases} \Rightarrow \cdots \Rightarrow \begin{cases} u_0 = \frac{1+p}{p^2} \\ u_1 = \frac{1}{p^2} \end{cases}$$