Ex 5.1

$$X \sim \exp(\lambda) \longrightarrow E[X] = \frac{1}{\lambda}$$

a) $$E[X] = \frac{1}{2} \Rightarrow \lambda = 2$$

On the other hand we know that : $P(X > t) = e^{-\lambda t}$

Hence $$P\left(X > \tfrac{1}{2}\right) = e^{-2 \times \frac{1}{2}} = \frac{1}{e}.$$

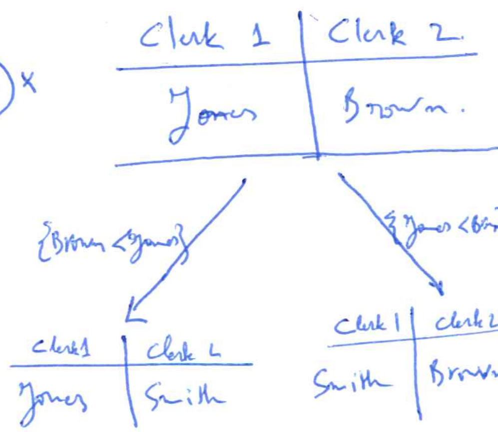
b) $$P\left(X > 12 + \tfrac{1}{2} \mid X > 12\right) \overset{(\text{Memory-lessness})}{=} P\left(X > \tfrac{1}{2}\right) = \frac{1}{e}.$$

Ex 5.2 By memorylessness, once we enter the bank the amount of time one spends in the bank is $\sum_{i=1}^{6} X_i$ where X_i is the service time for customer i. $X_i \sim \exp(\mu)$ $\forall i \in \{1 - 6\}$

Thus $$E\left[\sum_{i=1}^{6} X_i\right] \overset{(\text{indep.})}{=} \sum_{i=1}^{6} E[X_i] = \sum_{i=1}^{6} \frac{1}{\mu} = \frac{6}{\mu}.$$

Ex 5.6

$$P(\text{Smith is not last}) = P(\text{Smith} \overset{\text{Before}}{<} \text{Jones} \mid \text{Brown} < \text{Jones}) \times P(\text{Brown} < \text{Jones})$$
$$+ P(\text{Smith} < \text{Brown} \mid \text{Brown} > \text{Jones}) \times P(\text{Brown} > \text{Jones})$$
$$= \frac{\lambda_2}{\lambda_1 + \lambda_2} \times \frac{\lambda_2}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2} \times \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$



Ex 5.8

$$P(X \mid X < Y) = P(\min(X, Y))$$

but $\min(X, Y) \sim \exp(\lambda + \mu)$, thus :

$$P(X \mid X < Y) = (\mu + \lambda)\, e^{-(\mu + \lambda)x}$$

Ex 5.12

a) $$P(X_1 < X_2 < X_3) = P\left(\min\{X_1, X_2, X_3\} = X_1 \;\&\; \min\{X_2, X_3\} = X_2\right)$$
$$= P\left(\min\{X_2, X_3\} = X_2 \mid \min\{X_1, X_2, X_3\} = X_1\right) \cdot P\left(\min\{X_1, X_2, X_3\} = X_1\right)$$
$$\overset{\text{Memory-lessness}}{=} P\left(\min\{X_2, X_3\} = X_2\right) \cdot P\left(\min\{X_1, X_2, X_3\} = X_1\right)$$
$$= \frac{\lambda_2}{\lambda_2 + \lambda_3} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}$$

Ex 5.12 b) $P(X_1 < X_2 \mid \max_{1\le i\le 3}\{X_i\} = X_3) = \dfrac{P(X_1 < X_2 < X_3)}{P(\max_{1\le i\le 3}\{X_i\} = X_3)}$

$$= \frac{P(X_1<X_2<X_3)}{P(X_1<X_2<X_3) + P(X_2<X_1<X_3)}$$

$$= \frac{\frac{\lambda_2}{\lambda_2+\lambda_3}\cdot\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}}{\frac{\lambda_2}{\lambda_2+\lambda_3}\cdot\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} + \frac{\lambda_1}{\lambda_1+\lambda_3}\cdot\frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}}$$

$$= \frac{\frac{1}{\lambda_2+\lambda_3}}{\frac{1}{\lambda_2+\lambda_3}+\frac{1}{\lambda_1+\lambda_3}}$$

Exam 2009, Problem 1

a) * $X_n :=$ # of white balls. The transitioning of $(X_n)_{n\ge 0}$ from step n to $n+1$ is decided based solely on the composition of the urn at time n i.e. how many white balls there is among the total N balls.

Thus $P(X_{n+1}=j \mid (X_n, X_{n-1}, \dots, X_0) = (i_n, \dots, i_0)) = P(X_{n+1}=j \mid X_n = i_n)$

* X_n is homogeneous since the probabilities of transitioning are only dependent on the current state and the next one but not on the time step n.

b) * $S = \{0, 1, \dots, N\}$ * Only one class i.e. S'.

* Since $P^{(1)}_{0,0} > 0$ then the period is 1.
* Since $\mathrm{Card}(S') < \infty$ all states are (positive) recurrent.

c) $P_{i,j} \neq 0 \iff \begin{cases} i=j \Rightarrow P_{ii} = \frac{N-i}{N}(1-p) + \frac{i}{N}p \\ j=i+1 \Rightarrow P_{i,i+1} = \frac{N-i}{N}p \\ j=i-1 \Rightarrow P_{i,i-1} = \frac{i}{N}(1-p) \end{cases}$

d) $N=2 \Rightarrow P = \begin{bmatrix} (1-p) & p & 0 \\ \frac{1}{2}(1-p) & \frac{1}{2} & \frac{p}{2} \\ 0 & (1-p) & p \end{bmatrix}$. Thus: $\pi = \pi P \iff \begin{cases} \pi_0 = \pi_0(1-p) + \frac{\pi_1}{2}(1-p) \\ \pi_1 = \pi_0 p + \frac{\pi_1}{2} + \pi_2(1-p) \\ \pi_2 = \pi_1\frac{p}{2} + \pi_2 p \end{cases}$ (+) $\sum_{i=0}^{2}\pi_i = 1$

$\Rightarrow \pi = [(1-p)^2, 2p(1-p), p^2] = \left[\binom{2}{k}p^k(1-p)^{2-k}\right]_{k\in\{0,1,2\}}$.

Thus a legitimate guess for $N \ge 2$ is $\pi_k = \binom{N}{k}p^k(1-p)^{N-k}$.

e) let $T := \min\{n \ge 0 : X_n = N\}$ and $\alpha_i = E[T \mid X_0 = i]$. Then:

$$\alpha_i = 1 + P_{ii}\alpha_i + P_{i,i-1}\alpha_{(i-1)} + P_{i,i+1}\alpha_{i+1}$$

$(p=1) \Rightarrow \alpha_i = 1 + \frac{i}{N}\alpha_i + \frac{N-i}{N}\alpha_{i+1} \Rightarrow \alpha_i = \frac{N}{N-i} + \alpha_{i+1}$

$\Rightarrow \alpha_i = \sum_{j=i}^{N-1}\frac{N}{N-j}$ $(\alpha_{N-1+1} = \alpha_N = 0)$