Ex 5.15 We have $T = T_1 + T_2 + T_3 + T_4 + T_5$, where T_i is the time between $(i-1)^{th}$ and i^{th} failure. On the other hand:

$$E[T_1] = E[\min\{X_1, \dots, X_{100}\}] \quad \text{and} \quad \min\{X_1,\dots,X_{100}\} \sim_d \exp\left(\sum_{j=1}^{100} \frac{1}{200}\right)$$

$$\Rightarrow \quad E[T_1] = \frac{200}{\sum_{j=1}^{100} 1} = \frac{200}{100-i+1}$$

In general, we find: $E[T_i] = \dfrac{200}{100-i+1}$

Hence $E[T] = \sum_{i=1}^{5} \dfrac{200}{100-i+1}$ by independence of T_i's.

Similarly: $Var[T] \overset{(indep)}{=} \sum Var[T_i] = \sum_{i=1}^{5} \left(\dfrac{200}{100-i+1}\right)^2$

Ex 5.28 a) Let X_i be the random variable representing the life time of component i.

Then
$$P(\underbrace{\text{Component 1 is } 2^{nd} \text{ to fail}}_{=:A}) = P(A \cap \{\min\{X_1 - X_n\} = X_2) + \dots + P(A \cap \{\min\{X_1 - X_n\} = X_n\})$$

$$= \sum_{i=2}^{n} P(A \cap \{\min\{X_1,\dots,X_n\} = X_i\})$$

$$= \sum_{i=2}^{n} P(\underbrace{A \mid \{\min\{X_1,\dots,X_n\} = X_i\}}_{= \min\{\underbrace{X_1,\dots,X_n}_{\text{without } X_i}\} = X_1}) \times P(\min\{X_1,\dots,X_n\} = X_i)$$

$$= \sum_{i=2}^{n} \frac{\lambda_1}{\sum_{j\neq i} \lambda_i} \times \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$$

b) If T_i is the time between i^{th} & $(i-1)^{th}$ failures, then

$$E[\text{Time of } 2^{nd} \text{ failure}] = E[T_1 + T_2] = E[T_1] + E[T_2]$$

$$E[T_1] = \frac{1}{\sum_{i=1}^{n} \lambda_i} \qquad E[T_2] = \sum_{i=1}^{n} E[T_2 \mid \min\{X_1,\dots,X_n\} = X_i] \times P(\min\{X_1,\dots,X_n\} = X_i)$$

Thus
$$E[T_2] = \sum_{i=1}^{n} \frac{1}{\sum_{j\neq i} \lambda_j} \cdot \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$$

Hence:
$$E[\text{Time of } 2^{nd} \text{ failure}] = \frac{1}{\sum_{i=1}^{n} \lambda_i} + \sum_{i=1}^{n} \frac{1}{\sum_{j\neq i} \lambda_j} \cdot \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$$

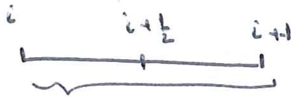
Ex 5.30

$$E[\underbrace{\text{Additional time for remaining pet}}_{=:T}] = E[T \mid \text{Cat died first}] \times P(\text{Cat died first}) + E[T \mid \text{Dog died first}] \times P(\text{Dog died first})$$

$$= \frac{1}{\lambda_d} \times \frac{\lambda_c}{\lambda_d + \lambda_c} + \frac{1}{\lambda_c} \times \frac{\lambda_d}{\lambda_d + \lambda_c}$$

Ex 5.32 a) Take $X := N(i+\frac{1}{2}) - N(i)$

$Y := N(i+1) - N(i+\frac{1}{2})$



$X(\omega)$ is somewhere in $[i, i+1]$

Then: $$P(X \leq 0, Y \leq 0) = P(X=0, Y=0) = 0$$

On the other hand: $$P(X \leq 0) = 1 - P(X>0) = 1 - \frac{(i+\frac{1}{2}) - i}{1-0} = \frac{1}{2}$$

Similarly $$P(Y \leq 0) = \frac{1}{2}$$

Thus $$P(X \leq 0, Y \leq 0) \neq P(X \leq 0) \cdot P(Y \leq 0)$$

$\Rightarrow$ dependence of increments.

b) Note that $N(t)$ can be written as follows: $N(t) = [t] + \mathbb{1}_{\{X \leq t - [t]\}}$

where $[t]$ is the integer part of t.

Then, for $T > 0$: $$N(t+T) - N(T) = [T+t] + \mathbb{1}_{\{X \leq T+t - [T+t]\}} - [T] - \mathbb{1}_{\{X \leq T - [T]\}}$$

$$\Rightarrow N(t+T) - N(T) = [t] + \mathbb{1}_{\{X \leq t - [t]\}} = N(t) - N(0)$$

Thus we have stationarity