Ex 5.46

$$\operatorname{Cov}\left(N(t), \sum_{i=1}^{N(t)} X_i\right) = E\left[N(t) \sum_{i=1}^{N(t)} X_i\right] - E[N(t)] \cdot E\left[\sum_{i=1}^{N(t)} X_i\right]$$

$$= E\left[E\left[N(t) \sum_{i=1}^{N(t)} X_i \mid N(t)\right]\right] - t\lambda \cdot E\left[E\left[\sum_{i=1}^{N(t)} X_i \mid N(t)\right]\right]$$

$$= E\left[N(t) \cdot N(t) \cdot E[X_i]\right] - t\lambda \cdot E\left[N(t) \cdot E[X_i]\right]$$

$$= \mu \cdot \left(\lambda t + (\lambda t)^2\right) - t\lambda \cdot \mu \cdot t\lambda$$

$$= \mu \lambda t.$$

Ex 5.50 X is given in fact by $X = N(T)$ with $T \sim \text{unif}([0,1])$

$\mu := E[T] = \frac{1}{2}$

$\sigma := \operatorname{Var}[T] = \frac{1}{12}$

Thus: $E[X] = \lambda\mu = \frac{7}{2}$

$$\operatorname{Var}[X] = \lambda\mu + \lambda^2\sigma^2 = \frac{7}{2} + \frac{49}{12} = \frac{91}{12}.$$

Ex 5.57 a) $P(N(t+1) - N(t) = 0) = \frac{\lambda^0}{0!} e^{-\lambda \cdot 1} = e^{-\lambda} = e^{-2}.$

b) $E[T_1 + T_2 + T_3 + T_4] + 12\text{ pm} = \frac{4}{\lambda} + 12\text{ pm} = 2 + 12\text{ pm}$

$= 14\text{ pm}.$

c) $$1 - P(N(t+2) - N(t) \leq 1) = 1 - \underbrace{e^{-2\lambda}}_{0 \text{ events}} - \underbrace{(2\lambda)e^{-2\lambda}}_{1 \text{ event}}.$$

Ex 5.64 a) By thm 5.2 in Ross, given $N(t) = n$, the events are uniformly distributed in the interval $[0,t]$. Let $\mathcal{U}_i \sim \text{Uniform}[0,t]$ be the arrival time of i-th person out of the n persons. Then:

$$E[X \mid N(t) = n] = E\left[\sum_{i=1}^{n} (t - \mathcal{U}_i)\right] \overset{\text{(indep)}}{=} n\, E[t - \mathcal{U}_i] = N(t) \cdot t - \frac{t}{2} = N(t) \cdot \frac{t}{2}$$

b) Similar to a), $\operatorname{Var}[X \mid N(t)] = N(t) \operatorname{Var}(t - \mathcal{U}_i) = N(t) \cdot \frac{t^2}{12}$

c) $$\operatorname{Var}(X) = E[\operatorname{Var}[X \mid N(t)]] + \operatorname{Var}[E[X \mid N(t)]] = \frac{t^2}{12} E[N(t)] + \frac{t^2}{2^2} \cdot \operatorname{Var}[N(t)]$$

$$= \frac{\lambda t^3}{3}.$$

Ex 5.67 Let I denote the class of satelites that have been launched before time s and are still operational at time t.

An event of this type is of the form $\{T > t-u,\ u<s\}$

where: i) u is the launch time.

ii) T is the lifetime.

Then by proposition 5.3 in Ross, $\{N_I(t)\}_{t \geq s}$ is a Poisson process with rate: $f(t) := \lambda \cdot \int_0^t P(T > t-u,\ u<s)\, du$

$$\Rightarrow \quad f(t) = \lambda \int_0^s P(T > t-u)\, du = \lambda \int_0^s \left(1 - G(t-u)\right) du$$

The probability we are seeking is $P(N_I(t) = 0) = e^{-\lambda \int_0^s (1-G(t-u))\,du}$.