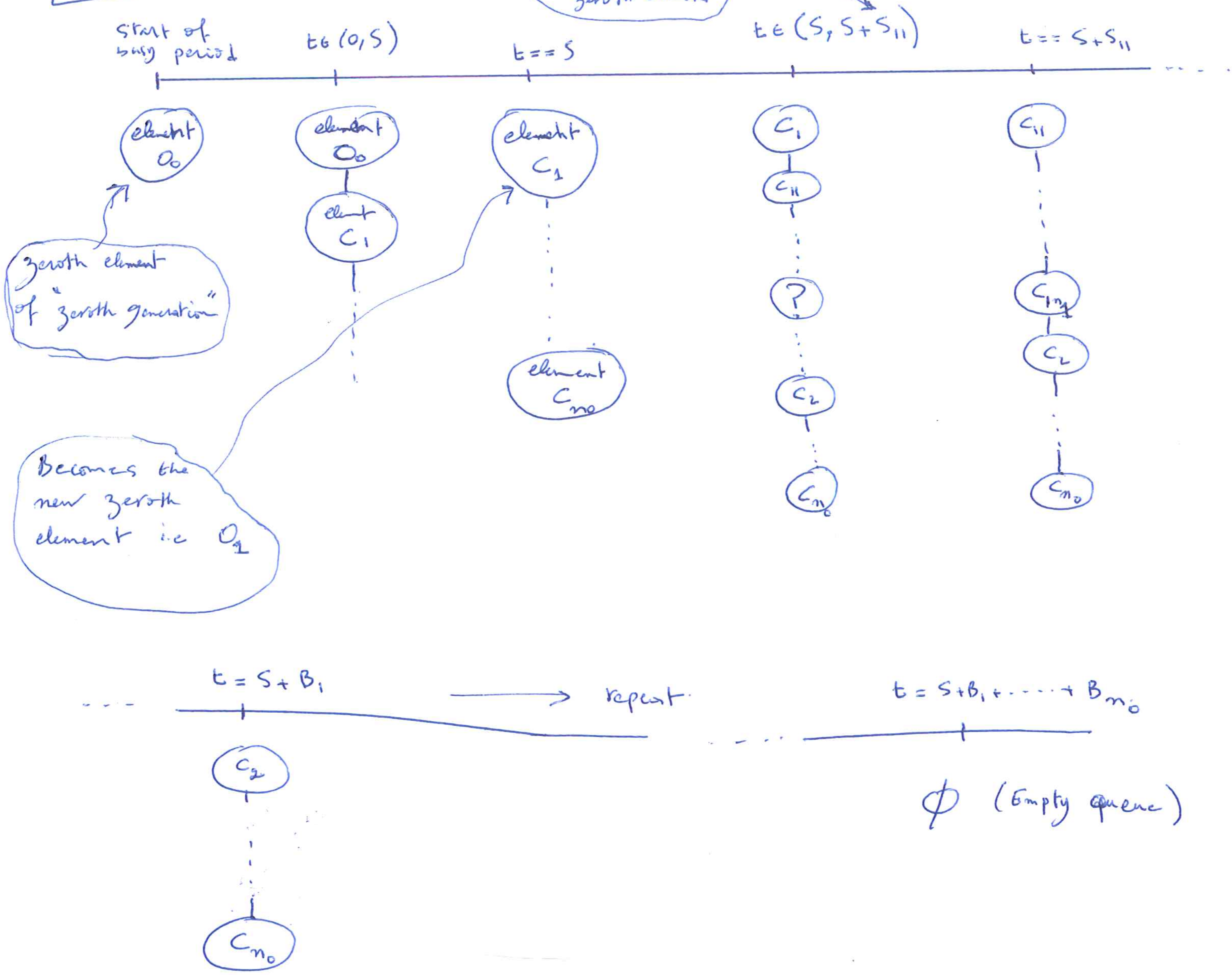
5.76 Suppose $N(S) = n_0$



Now applying the same logic, if we denote the number of customers during a busy period by M, we have:

$$M = 1 + M_1 + \cdots + M_{N(S)} = 1 + \sum_{i=1}^{N(S)} M_i$$

with $M \sim M_i \quad \forall i$.

Thus:

$$E[M] = E[E[M \mid S]] = E\left[1 + E\left[\sum_{i=1}^{N(S)} M_i \mid S\right]\right]$$

(eq. 5.24 Ross)

$$= 1 + E\left[\lambda S \, E[M_i]\right] = 1 + \lambda E[S] E[M_i]$$

$E[M_i] = E[M]$

$$\Rightarrow E[M] = \frac{1}{1 - \lambda E[S]} \quad (\text{if } 1 - \lambda E[S] > 0)$$

Similarly and by the variance formula for compound Poisson processes:

$$\mathrm{Var}(M) = E[\mathrm{Var}(M \mid S)] + \mathrm{Var}(E[M \mid S]) = E[S\lambda \, E[M_i^2]] + \mathrm{Var}(\lambda S \, E[M_i])$$

(eq 5.25 Ross)

$$\Rightarrow \quad \mathrm{Var}(M) = \lambda E[M^2]\, E[S] + \lambda^2 E[M]^2\, \mathrm{Var}(S)$$

$$= \lambda E[S]\left(\mathrm{Var}(M) + E[M]^2\right) + \lambda^2 E[M]^2\, \mathrm{Var}(S)$$

$$\Rightarrow \quad \mathrm{Var}(M) = \frac{\lambda E[S]\, E[M]^2 + \lambda^2\, \mathrm{Var}(S)\, E[M]^2}{1 - \lambda E[S]}.$$

Ex 5.78

$$\lambda(t) = \begin{cases} 0 & t \in [0,8) \\ 4 & t \in [8,10) \\ 8 & t \in [10,12) \\ 8+(t-12) & t \in [12,14) \\ 10-2(t-14) & t \in [14,17) \\ 0 & t \in [17,24) \end{cases}$$

$$\int_0^{24} \lambda(t)\,dt = \left(\int_0^8 + \int_8^{10} + \int_{10}^{12} + \int_{12}^{14} + \int_{14}^{17} + \int_{17}^{24} \right) \lambda(t)\,dt$$

$$= \dots = 63$$

Thus $N(24) - N(0) \sim \text{Poisson}(63)$.

Ex 6.1

Suppose $(N_1(t), N_2(t)) = (n_1, n_2)$ then:

$$v_{(n_1,n_2)} = \frac{1}{E[T]} \quad \text{where } T \sim \exp(n_1 n_2 \cdot \lambda)$$

This because by definition T is the amount of time the process spends in state (n_1, n_2), which is exponential with intensity proportional to the intensity of 1 mating multiplied by n_2 and then by n_1. Moreover

$$P_{(n_1,n_2),(n_1,n_2+1)} = P_{(n_1,n_2),(n_1+1,n_2)} = 1/2$$

Ex 6.2

Let $N_A(t)$, $N_B(t)$ denote the number of organisms in state A, B respectively.

Then if $(N_A(t), N_B(t)) = (n_A, n_B)$, we have:

$$v_{(n_A,n_B)} = [\text{Rate of } T, \text{ the time til the next event}]$$
$$= \alpha n_A + \beta n_B$$

On the other hand:

$$P_{(n_A,n_B)\to(n_A-1,n_B+1)} = P(\text{a division of type A happens before one of type B})$$
$$= P(T_A < T_B) \quad \text{where } \begin{cases} T_A \sim \exp(\alpha n_A) \\ T_B \sim \exp(\beta n_B) \end{cases}$$
$$= \frac{\alpha n_A}{\alpha n_A + \beta n_B}$$

Similarly:

$$P_{(n_A,n_B)\to(n_A+2,n_B-1)} = \frac{\beta n_B}{\alpha n_A + \beta n_B} = 1 - P_{(n_A,n_B)\to(n_A-1,n_B+1)}$$

Ex 6.3 Note that the failure rate depends on the machine and therefore a Birth - Death model isn't appropriate.

Still we can model the situation using a continuous MC with the following state space:

"$*$" := both are working

"$\dagger_2$" := both are down but 2 is been serviced.

$\dagger_1$:= " " " 1 is been serviced.

"$*_2$" := 1 is working, 2 is down & been serviced.

"$*_1$" := 2 ______, 1 ______

Now we have:

$$\nu_* = \mu_1 + \mu_2 \quad ; \quad \nu_{\dagger_2} = \nu_{\dagger_1} = \mu \quad ; \quad \nu_{*_2} = \mu_1 + \mu \quad ; \quad \nu_{*_1} = \mu_2 + \mu$$

Also:

$$P_{*,*_1} = P(1 \text{ fails before } 2) = \frac{\mu_1}{\mu_2 + \mu_1} = 1 - P_{*,*_2}$$

$$P_{*_1,\dagger_1} = \frac{\mu_2}{\mu_2 + \mu} = 1 - P_{*_1,*}$$

$$P_{*_2,\dagger_2} = \frac{\mu_1}{\mu_1 + \mu} = 1 - P_{*_2,*}$$

$$P_{\dagger_1 *_2} = P_{\dagger_2 *_1} = 1$$

(As $\dagger_1 \rightsquigarrow *_2$ is the only transition possible from $\dagger_1$. Similar for $\dagger_2$)

Ex 6.4 If we think of the people inside the queue as a population, then this population grows with rate $\lambda_n = \alpha_n \cdot \lambda$ and diminishes with rate $\mu_n = \mu$.