Ex 6.5

If we denote by $X(t)$ the # of infected members at t, then when $X(t) = n$, we have to find the intensity parameter controlling the transitions $X(t) = n \rightsquigarrow X(t+h) = n+1$.

A transition happens when a healthy member contacts an unhealthy one. This happens with probability $\frac{n(N-n)}{\binom{N}{2}}$. ($N$ is total pop. size)

This type of contacts happen (arrives) with intensity $\lambda \frac{n(N-n)}{\binom{N}{2}}$

The contacts that are infections happen then with intensity

$$p \lambda \frac{n(N-n)}{\binom{N}{2}}$$

1) This is true since only the number of infected members at the present moment is need to calculate probabilities of future transitions.

2) This is a pure birth process. (A pure death if we consider $N - X(t)$)

3) We are looking for $E[S_{N-1}]$ where $S_{N-1} := \sum_{i=1}^{N-1} T_i$ is the time when $N-1$ additional members become infected (starting from 1 individual).

T_i is the time between have i infected to $i+1$ infected.

As the above discussion shows $T_i \sim \exp\left(p\lambda \frac{i(N-i)}{\binom{N}{2}} \right)$

Thus
$$E[S_{N-1}] = E\left[\sum_{i=1}^{N-1} T_i\right] = \sum_{i=1}^{N-1} E[T_i]$$
$$= \sum_{i=1}^{N-1} \frac{\binom{N}{2}}{\lambda p \, i(N-i)} = \frac{1}{\lambda p} \binom{N}{2} \sum_{i=1}^{N-1} \frac{1}{i(N-i)}$$

Ex 6.6] We follow example 6.6 in [Ross].

a) Let $I_i = \begin{cases} 1 & \text{if the first transition out of } i \text{ is to } i+1. \\ 0 & \text{---------------------------------- } i-1. \end{cases}$

denote also by T_i the time it takes to go from i to $i+1$. Then:

$$(*)\begin{cases} E[T_i \mid I_i = 1] = \dfrac{1}{\lambda_i + \mu_i} \\ E[T_i \mid I_i = 0] = \dfrac{1}{\lambda_i + \mu_i} + E[T_{i-1}] + E[T_i] \end{cases}$$

Thus:

$$E[T_i] = E[T_i \mid I_i = 1] \cdot P(I_i = 1) + E[T_i \mid I_i = 0] \cdot P(I_i = 0)$$

$$= \frac{1}{\lambda_i + \mu_i} \cdot \frac{\lambda_i}{\lambda_i + \mu_i} + \frac{\mu_i}{(\lambda_i + \mu_i)^2} + \frac{\mu_i}{\lambda_i + \mu_i}\left(E[T_{i-1}] + E[T_i]\right)$$

$$\Rightarrow \quad E[T_i] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E[T_{i-1}].$$

On the other hand $E[T_0] = 1/\lambda_0$

Thus $E[T_i]$ can be computed for any $i \geq 0$, and the expected time to go from state 0 to 4 is just: $E[T_0] + \dots + E[T_3]$.

b) $E[T_2] + E[T_3] + E[T_4]$.

c) Using the fact that $(*)$ is equivalent to:

$$E[T_i \mid I_i] = \frac{1}{\lambda_i + \mu_i} + (1 - I_i)\left(E[T_{i-1}] + E[T_i]\right)$$

and that $I_i \sim \text{Bernoulli}\left(p = \frac{\lambda_i}{\lambda_i + \mu_i}\right)$, we get:

i) $$\operatorname{Var}(E[T_i \mid I_i]) = \operatorname{Var}(I_i)\left(E[T_{i-1}] + E[T_i]\right)^2$$

$$= \frac{\lambda_i}{\lambda_i + \mu_i} \cdot \frac{\mu_i}{\lambda_i + \mu_i}\left(E[T_{i-1}] + E[T_i]\right)^2$$

ii) * $\operatorname{Var}(T_i \mid I_i = 1) = \operatorname{Var}(X_i \mid I_i = 1) = \operatorname{Var}(X_i) = \dfrac{1}{(\lambda_i + \mu_i)^2}$

where the first equality says that, given $I_i = 1$, the time to reach $i+1$ is just the time $X_i \sim \operatorname{Exp}\left(\frac{1}{\lambda_i + \mu_i}\right)$ to make a transition out of i.
The 2nd equality says that the time until a transition is indep. of next state.

* $Var(T_i \mid I_i = 0) = Var(X_i + \text{time to get back to } i + \text{time to reach } i+1)$

(indep.) $= Var(X_i) + Var(T_i) + Var(T_{i-1})$.

Thus $E[Var(T_i \mid I_i)] = \dfrac{1}{(\mu_i+\lambda_i)^2} + \dfrac{\mu_i}{\mu_i+\lambda_i}\left(Var(T_{i-1}) + Var(T_i)\right)$

since $Var(T_i \mid I_i) = Var(X_i) + (1-I_i)\left(Var(T_{i-1}) + Var(T_i)\right)$

Finally using the conditional variance formula together with $Var(T_0) = \frac{1}{\lambda_0^2}$

we can compute $Var(T_i)$ for any $i \geq 0$.