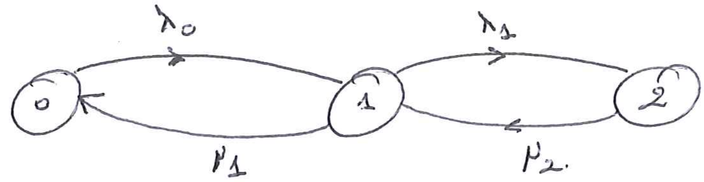
Ex 6.8



$\mu_1 = \mu_2 = \mu$

$\lambda_0 = 2\lambda$ and $\mu_n = \lambda_n = 0$ for all else.

$\lambda_1 = \lambda$

Then Thm 6.1 (~~or~~ example 6.10 directly) gives us all the backward Kolmogorov equations.

For example:
$$P'_{01}(t) = q_{01} \cdot P_{11}(t) - \nu_0 P_{01}(t)$$
$$\rightarrow \quad P'_{01}(t) = 2\lambda P_{11}(t) - 2\lambda P_{01}(t).$$

Ex 6.9

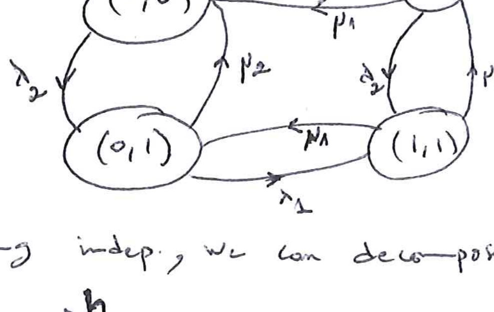
(with $\lambda_i = \lambda$).

Note that a pure birth process is a Poisson process, hence it is not difficult to see that, for a pure death process, we have:
$$P_{ij}(t) = \frac{1}{(i-j)!} e^{-\mu t} (\mu t)^{i-j}$$
provided $i \geq j > 0$.

The previous reasoning doesn't apply to P_{i0}, $i \geq 1$, since then the process can become negative!

Instead we use the fact that $\sum_{j \in S} P_{ij} = 1$ to get
$$P_{i0}(t) = 1 - \sum_{k=(i-1)}^{0} \frac{1}{(i-k)!} e^{-\mu t} (\mu t)^{i-k} = 1 - \sum_{k=0}^{(i-1)} \frac{1}{k!} e^{-\mu t} (\mu t)^k$$
$$\Rightarrow \quad P_{i0}(t) = \sum_{k=i}^{\infty} \frac{1}{k!} e^{-\mu t} (\mu t)^k$$

Ex 6.10



$\nu_{00} := \nu_{(0,0)} = \lambda_1 + \lambda_2$ $\qquad \lambda_{(0,0),(0,1)} = \lambda_1$.

$\nu_{01} := \nu_{(0,1)} = \lambda_1 + \mu_2$ $\qquad$ etc...

$\nu_{11} := \nu_{(1,1)} = \mu_1 + \mu_2$

$\nu_{10} := \nu_{(1,0)} = \lambda_2 + \mu_1$

* Using indep., we can decompose $P_{(i,j)(k,e)}(t)$ to $P^1_{i,k}(t) \times P^2_{j,e}(t)$.

where $P^h_{i,k}(t)$ is the probability that machine $n = 1, 2$ will be in k, at time t, starting from state i.

Thus each machine can be represented by a separate birth-death process.

Hence, using example 6.11, we get:

$$P^1_{0,0}(t) = \frac{1}{\lambda_1+\mu_1}\mu_1 + \frac{\lambda_1}{\lambda_1+\mu_1} e^{-(\lambda_1+\mu_1)t}$$

$$P^1_{1,0}(t) = \frac{1}{\lambda_1+\mu_1}\mu_1 - \frac{\mu_1}{\lambda_1+\mu_1} e^{-(\lambda_1+\mu_1)t}$$

By symmetry:

$$P^1_{1,1}(t) = \frac{1}{\lambda_1+\mu_1}\lambda_1 + \frac{\mu_1}{\lambda_1+\mu_1} e^{-(\lambda_1+\mu_1)t}$$

$$P^1_{0,1}(t) = \frac{1}{\lambda_1+\mu_1}\lambda_1 - \frac{\lambda_1}{\lambda_1+\lambda_1} e^{-(\lambda_1+\mu_1)t}.$$

Similar expression hold for $P^2_{k,j}(t)$ also and from this we can find, for instance, that:

$$P_{(0,0)(0,0)}(t) = P^1_{0,0}(t) \cdot P^2_{0,0}(t)$$

$$= \frac{1}{\lambda_1+\mu_1}\left(\mu_1 + \lambda_1 e^{-(\mu_1+\lambda_1)t}\right) \cdot \frac{1}{\lambda_2+\mu_2}\left(\mu_2 + \lambda_2 e^{-(\mu_2+\lambda_2)t}\right)$$

** Let's try to find the same expression "directly" from the original chain.

Kolmogorov's eq (Backward) $\Rightarrow$ $P'_{(0,0)(0,0)}(t) = -P_{(0,0)(0,0)}(t) \cdot (\lambda_1+\lambda_2) + \lambda_2 P_{(0,1)(0,0)}(t) + \lambda_1 P_{(1,0)(0,0)}(t)$

$$\Rightarrow P'_{(0,0)(0,0)}(t) = -P^1_{00}(t) P^2_{00}(t) (\lambda_1+\lambda_2) + \lambda_2 P^1_{00}(t) P^2_{10}(t) + \lambda_1 P^1_{10}(t) P^2_{00}(t)$$

$$= -\frac{(\lambda_1+\lambda_2)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}\left(\mu_1 + \lambda_1 e^{-(\mu_1+\lambda_1)t}\right)\left(\mu_2 + \lambda_2 e^{-(\mu_2+\lambda_2)t}\right)$$

$$+ \lambda_2 \cdot \frac{1}{(\lambda_1+\mu_1)}\left(\mu_1 + \lambda_1 e^{-(\mu_1+\lambda_1)t}\right) \frac{1}{\lambda_2+\mu_2}\left(\mu_2 - \mu_2 e^{-(\lambda_2+\mu_2)t}\right)$$

$$+ \lambda_1 \frac{1}{\lambda_1+\mu_1}\left(\mu_1 - \mu_1 e^{-(\lambda_1+\mu_1)t}\right) \frac{1}{\lambda_2+\mu_2}\left(\mu_2 + \lambda_2 e^{-(\lambda_2+\mu_2)t}\right)$$

$$= P^1_{00}(t)\left(-\frac{(\lambda_1+\lambda_2)}{(\lambda_2+\mu_2)}\mu_2 - \frac{(\lambda_1+\lambda_2)\lambda_2}{\lambda_2+\mu_2} e^{-(\lambda_2+\mu_2)t} + \frac{\lambda_2}{\lambda_2+\mu_2}\mu_2 - \frac{\mu_2\lambda_2}{\lambda_2+\mu_2} e^{-(\lambda_2+\mu_2)t}\right)$$

$$+ \lambda_1 P^1_{10}(t) P^2_{00}(t).$$

$$= P^1_{00}(t)\left(-\frac{\lambda_1\mu_2}{\lambda_2+\mu_2} - \frac{e^{-(\lambda_2+\mu_2)t}}{\lambda_2+\mu_2}\left((\lambda_1+\lambda_2)\lambda_2 + \mu_2\lambda_2\right)\right) + \lambda_1 P^1_{10}(t) P^2_{00}(t)$$

$$= P^1_{00}(t)\left(-\lambda_2 e^{-(\lambda_2+\mu_2)t}\right) + P^1_{00}(t)\left(-\frac{\lambda_1\mu_2}{\lambda_2+\mu_2} - \frac{e^{-(\lambda_2+\mu_2)t}}{(\lambda_2+\mu_2)}\left((\lambda_1+\lambda_2)\lambda_2+\mu_2\lambda_2-\lambda_2(\lambda_2+\mu_2)\right)\right)$$

$$+ \lambda_1 P^1_{10}(t)\, P^2_{00}(t).$$

$$= P^1_{00}(t)\left(\frac{d}{dt}P^2_{00}(t)\right) + P^1_{00}(t)\left(-\frac{\lambda_1\mu_2}{\lambda_2+\mu_2} - \frac{\lambda_1\lambda_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}\right)$$

$$+ P^2_{00}(t)\,\lambda_1\, P^1_{10}(t)$$

$$= P^1_{00}(t)\left(\frac{d}{dt}P^2_{00}(t)\right) + P^2_{00}(t)\left(\frac{\lambda_1\mu_1}{\lambda_1+\mu_1} - \frac{\lambda_1\mu_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}\right.$$

$$\left. - \frac{\lambda_1\mu_1}{\lambda_1+\mu_2} - \frac{\lambda_1\lambda_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}\right).$$

$$= P^1_{00}(t)\left(\frac{d}{dt}P^2_{00}(t)\right) + P^2_{00}(t)\cdot\underbrace{\left(-\lambda_1 e^{-(\lambda_1+\mu_1)t}\right)}_{=\frac{d}{dt}P^1_{00}}.$$

$$= P^1_{00}(t)\left(\frac{d}{dt}P^2_{00}(t)\right) + P^2_{00}(t)\left(\frac{d}{dt}P^1_{00}(t)\right).$$

$$= \frac{d}{dt}\left(P^1_{00}(t)\, P^2_{00}(t)\right) = \frac{d}{dt}P^{(t)}_{(0,0)(0,0)}$$

Thus the backward Kolmogorov equation holds in this case.

* Similarly the forward Kolmogorov eq can be checked.

Ex 5.11 a) Yule process (Example 6.8 in Book) is a pure birth process where each individual of the population gives birth with a rate λ. Hence:

$$T_{i-1} \sim_d \min(Y_1, \dots, Y_{i-1}) \quad \text{with} \quad Y_j \sim Exp(\lambda) \quad j\in\{1,\dots,i-1\}$$

$$\Rightarrow \quad T_{i-1} \sim Exp((i-1)\lambda) \quad \Rightarrow \quad T_i \sim Exp(i\lambda)$$

(T_i is the time to go from size i population to size $i+1$).

b) $\{\max(X_1, \dots, X_j) = X_K\} = \{X_{\sigma(1)} < X_{\sigma(2)} < \dots < X_{\sigma(j)} : \sigma\in \mathrm{perm}(j),\ \sigma(j)=K\}$

Thus $$X_K \overset{d}{=} X_{\sigma(j)} - X_{\sigma(j-1)} + (X_{\sigma(j-1)} - X_{\sigma(j-2)}) + \dots + (X_{\sigma(1)} - 0)$$

$$= \varepsilon_j + \dots + \varepsilon_1$$

where $\varepsilon_{(j-\ell+1)} \sim Exp(\ell\lambda)$

c) $P(T_1 + \cdots + T_j \le t) = P\left(\max_{1\le i\le j} X_i \le t\right) = \prod_{i=1}^{j} P(X_i \le t)$

$= (1 - e^{-\lambda t})^j$

d) $P_{1j}(t) = P(X(t) = j)$ $\quad$ (Given $X(0)=1$)

$= P(X(t) \geq j) - P(X(t) \geq j+1)$

$= P(T_1 + \cdots + T_j \le t) - P(T_1 + \cdots + T_{j+1} \le t)$

$= (1 - e^{-\lambda t})^{j-1} - (1 - e^{-\lambda t})^j = e^{-\lambda t}(1 - e^{-\lambda t})^{j-1}$

$= p(1-p)^{j-1}$

$\Rightarrow \quad X(t) \sim \text{Geometric}(p) \quad$ with $\quad p = e^{-\lambda t}$

e) Instead of starting with 1, we have now i individuals. Since they are indep, their distribution at time t is the sum of i indep. geometrics with parameters $(i, p = e^{-\lambda t})$.