Ex 7.1: We know that (See [Ross] page 411) that:

$$(N(t) \geq n) \iff (S_n \leq t)$$

Thus $(N(t) < n) \iff (S_n > t)$ (*)

Thus a) is true.

b) No. Assume $N(t) \leq n \iff S_n \geq t$, then:

$$P(N(t) \leq n) = P(S_n \geq t)$$. On the other hand:

$$P(N(t) \leq n) = P(N(t) < n) + P(N(t) = n)$$

(using (*)) $$= P(S_n > t) + P(S_n \leq t) - P(S_{n+1} \leq t)$$

$$= 1 - P(S_{n+1} \leq t) = P(S_{n+1} > t)$$

$\Rightarrow (\{S_n \geq t\} = \{S_{n+1} > t\}) \Rightarrow$ Contradiction (Since $\{S_{n+1} > t\} \not\Rightarrow \{S_n \geq t\}$)

c) No. Similar to b).

Ex 7.2: $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_i$ $X_i \sim Poi(\mu)$

a) Since the X_i's are indep. $S_n \sim Poi(n\mu)$

b) Use directly equation 7.3 on page 411 in [Ross] to get:

$$P(N(t) = n) = \sum_{i=0}^{[t]} \frac{e^{-n\mu}}{k!}(n\mu)^k - \sum_{i=0}^{[t]} \frac{e^{-(n+1)\mu}}{k!}((n+1)\mu)^k$$

Ex 7.3:

a) From the argument on page 412 in [Ross]

$$P(N(t) = n \mid S_n = y) = (1 - F(t-y)) \cdot \mathbb{1}_{\{t-y \geq 0\}}$$

b) This is essentially example 7.2 in [Ross]. Indeed:

$$P(N(t) = n) = \int_0^t \left(1 - \left(1 - e^{-\lambda(t-y)}\right)\right) \cdot \frac{\lambda e^{-\lambda y}(\lambda y)^{n-1}}{(n-1)!} dy$$

$$= \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \int_0^t y^{n-1} dy = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

$\Rightarrow N(t) \sim Poi(\mu t)$

Exam 2005, Problem 2: (Note the change from P_{ji} to our standard P_{ij})

Let $\quad P_{ij}(t) = P(X(t+u)=j \mid X(u)=i) \quad$ with

$$P_{ij}(h) = \begin{cases} \nu h + o(h) & ; \text{if } j=i+1 \\ i\mu h + o(h) & ; \text{if } j=i-1 \\ 1-(\nu+i\mu)h+o(h) & ; \text{if } j=i \\ o(h) & \text{else.} \end{cases}$$

a) $$q_{ii} = \lim_{h\to 0} \frac{P_{ii}(h)-P_{ii}(0)}{h} = \lim_{h\to 0} \frac{P_{ii}(h)-1}{h} = -(\nu+i\mu)$$

$$q_{ij} = \lim_{h\to 0} \frac{P_{ij}(h)}{h} = \begin{cases} \nu & ; \text{if } j=i+1 \\ i\mu & ; \text{if } j=i-1 \end{cases}$$

Thus

$$Q = [q_{ij}] = \begin{array}{c|ccccccccc} & 0 & 1 & 2 & 3 & \cdots & i-1 & i & i+1 & \cdots \\ \hline 0 & -\nu & \nu & 0 & 0 & & & & & \\ 1 & \mu & -(\nu+\mu) & \nu & 0 & & & & & \\ 2 & 0 & 2\mu & -(\nu+2\mu) & \nu & & & & & \\ 3 & & & & & & & & & \\ \vdots & & & & & & & & & \\ i-1 & & & & & & -(\nu+(i-1)\mu) & \nu & 0 & \\ i & & & & & & i\mu & -(\nu+i\mu) & \nu & \\ i+1 & & & & & & 0 & (i+1)\mu & -(\nu+(i+1)\mu) & \\ \vdots & & & & & & & & & \end{array}$$

b) π is stationary implies that $\pi Q = 0$

$$\Rightarrow \begin{cases} -\nu\pi_0 + \mu\pi_1 = 0 \\ \pi_{i-1}\nu - \pi_i(\nu+i\mu) + \pi_{i+1}(i+1)\mu = 0, \quad i \geq 1 \end{cases}$$

Take $i=1$, then we have:

$$\begin{cases} \pi_1 = \frac{\nu}{\mu}\pi_0. \\ \pi_0\nu - \pi_1(\nu+\mu) + \pi_2 2\mu = 0 \end{cases} \Rightarrow \begin{cases} \pi_1 = \frac{\nu}{\mu}\pi_0. \\ -\pi_1\nu + \pi_2 2\mu = 0 \end{cases} \Rightarrow \begin{cases} \pi_1 = \frac{\nu}{\mu}\pi_0 \\ \pi_2 = \frac{\nu}{2\mu}\pi = \frac{1}{2}\left(\frac{\nu}{\mu}\right)^2\pi_0 \end{cases}$$

Iterating, we find for all $i \geq 0$; $\quad \pi_i = \frac{1}{i!}\left(\frac{\nu}{\mu}\right)^i \pi_0$

Now, since $\sum_i \pi_i = 1$, we have $\quad \pi_0 + \sum_{i\geq 1} \frac{1}{i!}\left(\frac{\nu}{\mu}\right)^i \pi_0 = 1 \Rightarrow \pi_0 = \dfrac{1}{1+\sum_{i\geq 1}\frac{1}{i!}\left(\frac{\nu}{\mu}\right)^i}$

Hence $$\pi_i = \frac{1}{i!}\left(\frac{\nu}{\mu}\right)^i \cdot \frac{1}{1+\sum_{i\geq 1}\frac{1}{i!}\left(\frac{\nu}{\mu}\right)^i}.$$

c) See the proof of Kolmogorov's equations.