Ex 10.1 Let $s \le t$: $B(s) + B(t) = B(t) - B(s) + B(s) + B(s)$

Now $B(t) - B(s)$ is indep. of $B(s)$ and hence of $2B(s)$ as well.

On the other hand, $B(t) - B(s) \sim_d N(0, t-s)$ and $2B(s) \sim N(0, 4s)$

Thus, by the properties of Gaussian random variable, we get :

$$B(s) + B(t) \sim_d N(0, t-s+4s) .$$

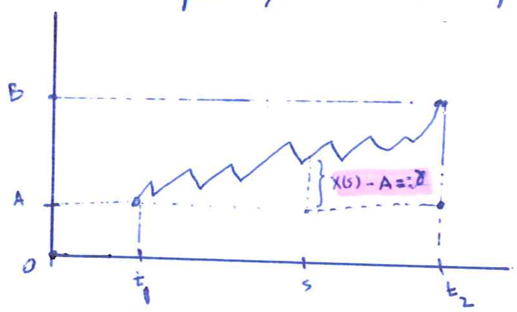
Ex 10.2 Let $0 < t_1 < s < t_2$, what is the dist. of $\left(B(s) \,\middle|\, \begin{matrix} B(t_1) = A \\ B(t_2) = B \end{matrix}\right)$?

Equation (10.4) on page 610 in [ROSS] says that : Change $B(t) \to X(t)$

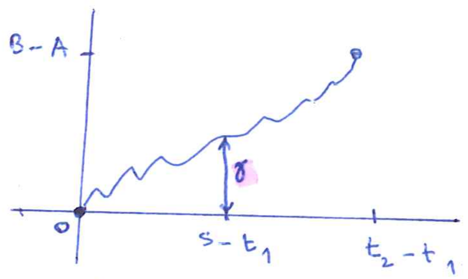
$$(*) \quad \left(X(s) \mid X(t) = \beta\right) \sim_d N\left(\frac{s}{t}\beta, \frac{s}{t}(t-s)\right) \quad \text{for } s<t .$$

To be able to use this result, we use some of the properties of Brownian motion.

In what follows, think of the following:



$\Longrightarrow$



Consider the shifted random variable $X(s) - A$, then:

$$\left(X(s) - A \,\middle|\, \begin{matrix} X(t_1) = A \\ X(t_2) = B \end{matrix}\right) \sim_d \left(X(s-t_1) \,\middle|\, \begin{matrix} X(t_1 - t_1) = A - A \\ X(t_2 - t_1) = B - A \end{matrix}\right)$$

$$\sim_d \left(X(s-t_1) \,\middle|\, \begin{matrix} X(0) = 0 \\ X(t_2 - t_1) = B - A \end{matrix}\right)$$

$$\sim_d N\left(\frac{s-t_1}{t_2-t_1}(B-A), \frac{s-t_1}{t_2-t_1} \cdot (t_2 - s)\right) \quad \text{(By using } (*) \text{)}$$

$$\Rightarrow \quad X(s) \sim_d N\left(A + \frac{s-t_1}{t_2-t_1}(B-A), \frac{s-t_1}{t_2-t_1}(t_2-s)\right)$$

Ex 10.4:

a) $\{T_a < \infty\} = \bigcup_{n \ge 1} \{T_a \le n\} \Rightarrow P(T_a < \infty) = \lim_{n\to\infty} P(T_a \le n) = \frac{2}{\sqrt{2\pi}} \lim_{n\to\infty} \int_{\frac{|a|}{n}}^{\infty} e^{-\frac{y^2}{2}} dy$

$$\Rightarrow P(T_a < \infty) = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-y^2/2} dy = 2\Phi(0) = 1$$

Φ ← CDF of Gaussian.

b) $f_{T_a}(t) = \frac{\partial}{\partial t} P(T_a \le t) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{|a|}{\sqrt{t}}\right)^2} \cdot \frac{d}{dt} \frac{|a|}{\sqrt{t}}$ (Leibniz's rule)

$$= \frac{|a|}{\sqrt{2\pi}} \frac{1}{t^{3/2}} e^{-\frac{a^2}{2t}} .$$

Hence: $$E[T_a] = \frac{|a|}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{t}} e^{-\frac{a^2}{2t}} dt \geqslant \frac{|a|}{\sqrt{2\pi}} \int_A^\infty \frac{e^{-\frac{a^2}{2t}}}{\sqrt{t}} dt$$

for any $A > 0$.

Now we use a comparison test, namely since $e^{-\frac{a^2}{2t}} = 1 - \frac{a^2}{2t} + O\left(\frac{1}{\sqrt{t}}\right)$

we have $$\lim_{t \to \infty} \frac{\frac{1}{\sqrt{t}} e^{-\frac{a^2}{2t}}}{\frac{1}{\sqrt{t}}} = 1$$

But $\int_A^\infty \frac{1}{\sqrt{t}} dt$ diverges, and thus $\int_A^\infty \frac{e^{-\frac{a^2}{2t}}}{\sqrt{t}} dt$ must also diverge

Hence $$E[T_A] = \infty .$$

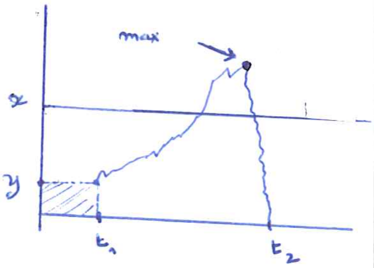
Ex 10.7 | We know that $P\left(\max_{0 \le s \le t} X(s) \geq a\right) = 2P(X(t) \geq a)$

for Brownian motion s.t $X(0) = 0$.

Now: $$P\left(\max_{t_1 \le s \le t_2} X(s) > x\right) = \int_{\mathbb{R}} P\left(\max_{t_1 \le s \le t_2} X(s) > x \mid X(t_1) = y\right) \frac{1}{\sqrt{2\pi t_1}} e^{-\frac{y^2}{2t_1}} dy$$

Now, if $y \geq x$ then $P\left(\max_{[t_1,t_2]} X(s) > x \mid X(t_1) = y\right) = 1$ (I)

If $y < x$, we can use the shift we used in Ex 10.2 to get:



$$P\left(\max_{[t_1,t_2]} X(s) > x \mid X(t_1) = y\right) = P\left(\max_{[0, t_2 - t_1]} X(s) > x - y\right)$$
$$= 2P\left(X(t_2 - t_1) > x - y\right) \quad \text{(II)}$$

Now (I) + (II) give the result.

Ex 10.9 | $X(s) \sim N(\mu s, \sigma^2 s)$, $X(t) \sim N(\mu t, \sigma^2 t)$ thus $\begin{bmatrix} X(s) \\ X(t) \end{bmatrix} \sim N_2(\Lambda, \Sigma)$

where $\Lambda = \begin{bmatrix} \mu s \\ \mu t \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sigma^2 s & Cov(X(s), X(t)) \\ \cdot & \sigma^2 t \end{bmatrix}$

Now $cov(X(s), X(t)) = E[(X(s) - \mu s)(X(t) - \mu t)] = E[X(s) \cdot X(t)] - E[X(s)] \cdot E[X(t)]$

$= E[(X(t) - X(s) + X(s)) X(s)] - \mu s \, \mu t$

$= E[(X(t) - X(s)) X(s)] + \underbrace{E[X(s)^2]} - \mu s \, \mu t$

(indep. increments) $= \underbrace{E[X(t) - X(s)] \cdot E[X(s)]} + \sigma^2 s + (\mu s)^2 - \mu^2 s t$

$= (t - s)\mu \cdot \mu s + \sigma^2 s + (\mu s)^2 - \mu^2 s t = \sigma^2 s \Rightarrow \Sigma = \begin{bmatrix} \sigma^2 s & \sigma^2 s \\ \sigma^2 s & \sigma^2 t \end{bmatrix}$

Exam 2007, Problem 4:

a) Similar t Ex 10.9.

b) • $Z(t) := \frac{W(t)}{\sqrt{t}}$ $\Rightarrow$ $Var(Z(t)) = \frac{1}{t} Var(W(t)) = \frac{1}{t}\sigma^2 t = \sigma^2$.

• $Cov(Z(s), Z(t)) = \frac{1}{\sqrt{s \cdot t}} \cdot Cov(W(s), W(t)) = \frac{1}{\sqrt{s \cdot t}} \sigma^2 s = \sigma^2 \cdot \frac{\sqrt{s}}{\sqrt{t}}$

$\Rightarrow$ $Cor(Z(s), Z(t)) = \frac{\sigma^2 \frac{\sqrt{s}}{\sqrt{t}}}{\sqrt{\sigma^2} \cdot \sqrt{\sigma^2}} = \sqrt{\frac{s}{t}}$.