Problem 1:

a) i) * State 0 & 1 communicate with each other, thus they belong to the same class. Call it C_1.

* Same for 2 & 3. Call it C_2

* 4 communicates only with itself. Call this class C_3.

ii) Once in classes C_2 or C_3 leaving is not possible. Thus they are recurrent.

From C_1 one can reach C_2 or C_3 and never come back. Thus it is transient.

b) $$P^4 = P \times P \times P \times P = \left[\; \cdots \; \right]$$

$[P^4]_{1,5} = P(X_4 = 4 \mid X_0 = 0)$ i.e the conditional probability of reaching state 4 in four steps starting from state 0.

c) Introduce a modified MC, call it W_n, having the following transition matrix Q given by:

Note: A stands for $\{2,3,4\}$

$$Q_{ij} = P_{ij} \qquad i,j \in \{0,1\}.$$

$$Q_{iA} = P_{i2} + P_{i3} + P_{i4} \qquad i,j \in \{0,1\}$$

$$Q_{AA} = 1$$

Thus $$Q = \begin{bmatrix} 0 & \frac{1}{4} & 3/4 \\ \frac{1}{4} & 0 & 3/4 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $P(X_4 = 4,\ X_k \notin \{2,3,4\} \text{ for } k = 1,2,3 \mid X_0 = 0)$

$$= Q^3_{00} P_{04} + Q^3_{01} P_{14} = 0$$

Alternatively, one can argue using a) that the event $\{X_4 = 4, X_k \notin \{2,3,4\}$ for $k = 1,2,3\}$ ~~happens or~~ is equivalent, when $\{X_0 = 0\}$, to $\{X_0 = 0, X_1 = 1, X_2 = 0, X_3 = 1, X_4 = 4\}$

Thus $P(X_4 = 4,\ X_k \notin \{2,3,4\} \text{ for } k = 1,2,3 \mid X_0 = 0) = P_{01} P_{10} P_{01} \cdot P_{14} = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot 0 = 0$

d) Using a similar argument as in c), we have:

$$P(X_4 = 0,\ X_k \notin \{2,3,4\},\ k = 1,2,3 \mid X_0 = 0) = P_{01} P_{10} P_{01} P_{10} = \frac{1}{16^2} = \frac{1}{256}$$

e) Let $\mu_i = E[T \mid X_0 = i]$ $i \in S$ and $T = \min\{n \geq 0 : X_n \in \{2,3,4\}\}$

Then if one starts in $\{2,3,4\}$, the time it takes to reach $\{2,3,4\}$ is zero. Thus $\mu_i = 0$ $\forall\ i \in \{2,3,4\}$.

On the other hand: $\mu_0 = E[T \mid X_0 = 0] = 1 + P_{01}\mu_1 + \cancel{P_{00}}\mu_0$

$$\Rightarrow \boxed{\mu_0 = 1 + \frac{1}{4}\mu_1}$$

similarly $\mu_1 = E[T \mid X_0 = 1] = 1 + \cancel{P_{11}\mu_1} + P_{10}\mu_0.$

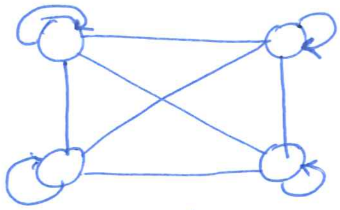
$$\Rightarrow \boxed{\mu_1 = 1 + \frac{1}{4}\mu_0}$$

f) We know that $S = (I - P_T)^{-1}$

where $P_T = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{bmatrix}$. Thus: $S = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{bmatrix} \right)^{-1}$

$$\Rightarrow S = \begin{bmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{4} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 16/15 & 4/15 \\ 4/15 & 16/15 \end{bmatrix}$$

Problem 2 :



a) Since all the modes have 4 connections, a uniform distribution over these connections implies that: $q(i,j) = \frac{1}{4} \quad \forall\, i,j \in \{1,2,3,4\}$

b) We know that such a MC can be constructed and it should have transitions:

$$\begin{cases} P_{ij} = \alpha(i,j)\, q(i,j) \\ P_{i,i} = q(i,i) + \sum_{k \neq i} q(i,k)\,(1 - \alpha(i,k)) \end{cases}$$

where $\alpha(i,j) = \min\left(\frac{\pi_j}{\pi_i} \frac{q(j,i)}{q(i,j)}, 1 \right)$ is the acceptance rate.

Using these ingredients we find that:

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{16} & \frac{2}{16} & \frac{3}{16} & \frac{10}{16} \end{bmatrix}$$

c) $Q = [q(i,j)]_{i,j}$ becomes $\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

d) Similar to b) we get

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{2}{9} & \frac{4}{9} & \frac{1}{3} & 0 \\ \frac{1}{12} & \frac{1}{8} & \frac{13}{24} & \frac{1}{4} \\ \frac{1}{9} & 0 & \frac{1}{3} & \frac{5}{9} \end{bmatrix}$$