Poisson process : 3 definitions.

These 3 def are based on the following 3 approaches:

Approach 1: (Point process / Global perspective)

a) # of events in an interval I is $\sim$ Poisson $(\lambda(I))$

where $\lambda(I)$ is a function of the interval I. e.g $\lambda(I) = \lambda \times \text{size}(I)$.

b) # of events in disjoint intervals are independent.

Approach 2: (Interarrival times + Jump chains)

Let S_n be the time when the n-th event happens. $(0 < S_1 < S_2 \dots)$

Define $T_n := S_n - S_{n-1}$ $\quad S_0 := 0$

↑ The inter arrival times.

Then $T_n \sim \exp(\lambda)$ $\quad \forall n \geq 1$.

Approach 3: (Local / infinitesimal behavior)

a) $P(N(t+h) - N(t) = 1 \mid N(s), 0 \leq s \leq t) = \lambda h + o(h)$

b) $P(N(t+h) - N(t) \geq 2 \mid N(s), 0 \leq s \leq t) = o(h)$.

[c) Indep over disjoint intervals]

Def 1: Let $\lambda > 0$. The counting process $\{N(t)\}_{t \geq 0}$ is called Poisson with rate λ if:

1) $N(0) = 0$

2) $N(t)$ has indep increments

3) # of events in any interval of length $T > 0$ has Poisson $(\lambda . T)$ distribution

Def 2:

Let $\lambda > 0$. The counting process $\{N(t)\}_{t \geq 0}$ is called Poisson with rate λ if:

1) $N(0) = 0$

2) $N(t)$ has independent increments

3) • $P(N(t+h) - N(t) = 1) = \lambda h + o(h)$.

• $P(N(t+h) - N(t) \geq 2) = o(h)$.

Def 3: Let $\lambda > 0$. The counting process $\{N(t)\}_{t \geq 0}$ is called Poisson with rate λ if:

$N(t) = n$ when $S_n \leq t \leq S_{n+1}$

with $S_n := \sum_{i=0}^{n} T_i$ and $T_0 := 0$

for a sequence of indep. exponentially distributed T_i's with mean $\frac{1}{\lambda}$