STK2130-sp19: mandatory assignment

Posted: Thursday March 7th. at 9 am
Deadline: Thursday March 28th at 2.30 pm.

This is text for the assignments in STK2130-sp19. It consists of two problems. Both handwritten reports and answers using a word processing system are acceptable.

Cooperation and discussing with other students are permitted and encouraged, but each student must deliver a separate and individually formulated report, There is more information at:
http://www.uio.no/english/studies/examinations/compulsory-activities/mn-math-mandatory.html

Enclose the parts of the computer outputs which are necessary for answering the questions. The other parts can be collected in appendices. When you refer to material in these, be careful to indicate explicitly where. You can use the computer language at your choice, pyton, R, matlab etc.

The assignments must be uploaded using the electronic filing system, Canvas.

Problem 1

A Markov chain $X_n, n = 0, 1, \dots$ has states $\{0, 1, 2, 3, 4\}$ and transiton matrix

$$P = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0. & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

a) The chain has classes, $C_1 = \{0, 1\}$, $C_2 = \{2, 3\}$ and $C_2 = \{4\}$. Explain why there are three classes and C_1 is transient. Are the others recurrent?

b) Compute P^4. What does the matrix element $(1, 5)$ mean, i.e the element in the upper right corner?

c) Compute $P(X_4 = 4,\ X_k \notin \{2, 3, 4\},\ k = 1, 2, 3|X_0 = 0)$. Explain the result.

d) Compute $P(X_4 = 0,\ X_k \notin \{2, 3, 4\},\ k = 1, 2, 3|X_0 = 0)$.

e) Let $T = \min\{n \geq 0 : X_n \in \{2, 3, 4\}$, i.e. the first period the chain is in $\{2, 3, 4\}$ and let $\mu_i = E[T|X_0 = i],\ i = 0, 1, 2, 3, 4$.

   Explain why $\mu_2 = \mu_3 = \mu_4 = 0$ and μ_0 and μ_1 satisfy the equations

$$\mu_0 = \frac{1}{4}\mu_1 + 1$$
$$\mu_1 = \frac{1}{4}\mu_0 + 1$$

Find μ_0 and μ_1.

f) Let $s_{ij}, i, j = 0, 1$ be the expected number of visits to state 0 and 1 conditional on $X_0 = i$. Find and solve the equations for determining s_{ij}.

Problem 2

Graphs can be helpful when setting up a MCMC chain. We will consider some examples. Let $\mathcal{G}$ be a graph with k nodes, $i = 1, \ldots, k$ where each node is connected to d_i others $i = 1, \ldots, k$. Let the proposal distribution be

$$q(i,j) = \begin{cases} \frac{1}{d_i} & \text{if nodes i and j are connected} \\ 0 & \text{else} \end{cases}$$

Consider a graph with four nodes, and the distribution $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ where $\pi_1 = 0.1, \pi_2 = 0.2, \pi_3 = 0.3, \pi_4 = 0.4$. The aim is to construct Markov chains having π as the stationary distribution.

a) Consider first the graph where each of the four nodes nodes is connected to itself and all the neighbors, i.e. each node has four connections. Use the Hastings-Metropolis algorithm to construct a Markov chain with proposal distribution described by the graph. Run the chain 10000 periods, and evaluate $E[\log(X)]$.

b) Find the transition matrix for the Markov chain constructed in part a).

c) Next consider the modified graph from part a) where the nodes 2 and 4 have no connection. Use the Hastings-Metropolis algorithm to construct a Markov chain with proposal distribution described by this graph. Run the chain 10000 periods, and evaluate $E[\log(X)]$.

d) Find also the transition matrix for the Markov chain from part c).