Exercise 3, Exam 2010

A PROCESS $X_0, X_1, X_2, \ldots$ takes values on $\{0,1,2\}$ and we will suppose that it behaves as a Markov process with transition probability matrix

$$P = \begin{pmatrix} 1-p & p & 0 \\ 1-p & 0 & p \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix},$$

where p is a parameter in $(0,1)$.

(a) Justify whether the states are recurrent or transient. What is the period of this Markov chain?

(b) Compute the probability $P^{(2)}_{0,0}$ that $X_2 = 0$ given that $X_0 = 0$.

(c) Find the probabilities

$$P(X_{88} = j | X_{87} = 0, X_{89} = 0) \text{ for } j = 0, 1, 2.$$

(d) Write down the equations to determine (π_0, π_1, π_2), the Markov chain's limiting probabilities. These equations are not difficult to solve, but you do not need to do this within exam's scheduled time; the solution is

$$(\pi_0, \pi_1, \pi_2) = \left(\frac{2-p}{2+p+3p^2}, \frac{2p}{2+p+3p^2}, \frac{3p^2}{2+p+3p^2} \right).$$

(e) Assume that the chain starts at $X_0 = 0$, and let T_0 be the time spent before the chain attains 0 again. What is the expected value of T_0?

(f) Let W denote the time when the Markov chain visits state 2 for the first time, that is, $\min\{n \geqslant 0 : \ X_n = 2\}$ and define

$$u_i = E[W | X_0 = i] \text{ for } i = 0, 1, 2.$$

What is u_2? Derive the equations to determine u_0 and u_1 and solve these equations if you have time.