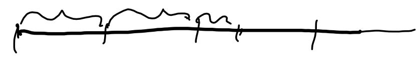

$N(t)$ counting process counting events in $(0,t]$

Poisson process rate λ

(i) $N(0)=0$

(ii) $\{N(t), t\ge 0\}$ independent increments

(iii) $P(N(t+h)-N(t)=1)=\lambda \cdot h + o(h)$

(iv) $P(N(t+h)-N(t)\ge 2)=o(h)$

Define the random variables $T_1, T_2, \dots$ as follows. T_1 is the time of first event, and T_n is the time between the $(n-1)$th an nth event

$\{T_n, n\ge 1\}$ is called the sequence of interarrival times

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Distribution of T_n ?

$\{T_1 > t\} \Leftrightarrow \{\text{no event in } (0,t]\}$

so $P(T_1 > t) = \frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t}$

So $T_1 \sim \lambda \exp(-\lambda t)$

Also $P(T_2 > t) = E[P(T_2 > t \mid T_1)]$

$P(T_2 > t \mid T_1 = s) = P(\text{no event in } (s, s+t] \mid T_1 = s)$

$= P(\text{no events in } (s, s+t])$

by independent increments

$= e^{-\lambda t}$

by stationary increments

Hence $T_2 \sim \lambda \exp(-\lambda t)$

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Continuing the argument

Proposition 5.1 The interarrival times $T_n, n=1, \dots$ are i.i.d $T_i \sim \lambda \exp(-\lambda x)$

Remark 5.3.3 The assumption of stationarity and independent increments essentially means that the process probabilistically restarts itself at any point in time. Independent increments means that the process has no memory and stationary increments implies that the distributions remain the same

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Let $S_n = T_1 + \dots + T_n$, S_n is ~~taken~~ often called the <u>waiting time</u> until the nth event

We know that $S_n \sim \text{gamma}(n, \lambda)$

Alternative derivation 1.

$N(t) \ge n \Leftrightarrow S_n \le t$

nth event before or at t ⟺ n or more events before or at t.

Hence, $F_{S_n}(t) = P(N(t)\ge n) = \sum_{j=n}^{\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t}$

Differentiating w.r.t. λ.

$$f_{S_n}(t) = -\sum_{j=n}^{\infty} \lambda \cdot \frac{(\lambda t)^j}{j!} e^{-\lambda t} + \sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^{j-1}}{(j-1)!}$$

$$= \underline{\lambda \cdot e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}} + \underbrace{\sum_{j=n+1}^{\infty} \lambda \cdot e^{-\lambda t} \frac{(\lambda t)^{j-1}}{(j-1)!} - \sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^j}{j!}}_{0.}$$

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Alternative derivation 2.

$$P(t < S_n < t+h) = \sum_{j=0}^{\infty} P(N(t)=n-1, j \text{ events in } (t,t+h))$$

$$= P(N(t)=n-1, \text{one event in } (t,t+h)) + o(h)$$

$$= P(N(t)=n-1) P(\text{one event } (t,t+h)) + o(h)$$

$$= \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t} [\lambda h + o(h)] + o(h)$$

$$= \lambda \cdot e^{-\lambda t} \cdot \frac{(\lambda t)^{n-1}}{(n-1)!} \cdot h + o(h)$$

$$\frac{F_{S_n}(t+h) - F_{S_n}(t)}{h} = \frac{\lambda^n}{(n-1)!} \cdot t^{n-1} e^{-\lambda t} + \frac{o(h)}{h}$$

Let $h \to 0$, then $f_{S_n}(t) = \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t}$.

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Remark 5.3.4 The Poisson process can alternatively be defined by a sequence $\{T_n, n\ge 1\}$ of i.i.d variables $T_i \sim \lambda \cdot \exp(-\lambda x)$

Define the counting process by

$N(t) = \max\{n : S_n \le t\}$, $S_0 = 0$

i.e that the nth event occurs at time S_n.

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Example Airplanes arriving according to Poisson process rate λ

interarrival times mean $\frac{1}{\lambda} = 5$ min

(i) $E[$ 10'th arrival $] = E[S_{10}] = \frac{10}{\lambda} = 50$ min.

(ii) $P($ more than 10 minutes between 10 th and 11th arrival $)$

$= P(T_{11} > 10) = e^{-10\lambda} = e^{-2} \approx 0.135.$

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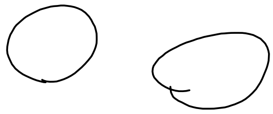
Poisson process in plane



The number of events in two areas, i.e the number of point in areas are independent.

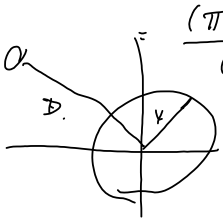
The number also depends on size of area. The number of events in an area is Poiss distributed where the parameter is product of rate and size of area.

mar. 4-11:58

D distance from O to closest point

$P(D > x) = P($ no points in circle with radius x, center O $)$

$= \frac{(\pi x^2 \cdot \lambda)^0}{0!} \cdot e^{-\lambda \pi x^2}$



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