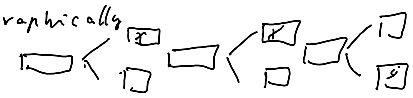
Consider a Poisson process with rate λ and let each event be classified as a type I or type II event with probability p and $1-p$ respectively. The classification is independent of the events in the Poisson process.

Let $N_1(t)$ and $N_2(t)$ count the number of type I and type II customers.

Graphically



Proposition 5.2. $\{N_1(t), t\geq 0\}$ and $\{N_2(t), t\geq 0\}$ are both Poisson processes having rates $p\lambda$ and $(1-p)\lambda$ respectively. The two processes are independent.

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Proof. Will verify that $\{N_1(t), t\geq 0\}$ is a Poisson process with rate λp by checking the definition of a Poisson process.

(i) $N_1(0)=0$ since $N(0)=N_1(0)+N_2(0)$

(ii) $N_1(t)$ has stationary and independent increments: Conditioned on the number of events in an interval, one find the distribution of type I events. This will only depend on the length of the interval and is independent of what happens in other intervals.

(iii)
$$P(N_1(h)=1) = P(N_1(h)=1 \mid N(h)=1)\,P(N(h)=1) + P(N_1(h)=1 \mid N(h)\geq 2)\,P(N(h)\geq 2)$$
$$= p\cdot[\lambda h + o(h)] + o(h) = \lambda p h + o(h)$$

(iv) $P(N_1(h)\geq 2) \leq P(N(h)\geq 2) = o(h)$

Similarly, $\{N_2(t), t\geq 0\}$ is a Poisson process with rate $(1-p)\lambda$

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Independence? Events of type I in an interval is independent of events in non-overlapping intervals, and therefore also of whether the events are of type I or II.

Example: Call to a bank, 30 calls per hour
type I: questions about accounts 80%
type II: — " — " — loans

The number of questions about loan in 3 hours is Poisson distributed with $3\cdot 30/5 = 18$

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Example Company strategy for accepting offers

- Offers arrive as a Poisson process, rate λ
- Value of random offer, $X\sim f(x)$, c.d.f $F(x)$
- Strategy: Accept or reject and wait for new offer
- cost of rate c for unsold items
- accept if offer large than critical value y

Value of y maximizing return?
Maximal expected return?

Fix y, offers greater than y arrive at rate $\lambda(1-F(y))$ (Proposition 5.2)

$R(y)$ return.

$$E[R(y)] = E[\text{accepting offer}] - c\,E[\text{time to accept}]$$
$$= E[X \mid X>y] - c\cdot\frac{1}{\lambda(1-F(y))}$$

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$$= \int_0^\infty x\, f_{X\mid X>y}(x)\,dx - \frac{c}{\lambda[1-F(y)]}$$
$$= \int_y^\infty \frac{x\cdot f(x)}{[1-F(y)]}\,dx - \frac{c}{\lambda[1-F(y)]} = \overbrace{\frac{1}{[1-F(y)]}\left[\int_y^\infty x f(x)\,dx - \frac{c}{\lambda}\right]}$$

First order conditions

$$\frac{\partial}{\partial y}R(y) = \frac{(-1)}{[1-F(y)]^2}\cdot(-f(y))\Big[\underbrace{\int_y^\infty x f(x)\,dx}_{-c/\lambda}\Big] - \frac{1}{[1-F(y)]}\cdot y f(y) = 0$$

or $\quad y[1-F(y)] = \int_y^\infty x f(x)\,dx - \frac{c}{\lambda}$

$\quad y\int_y^\infty f(x)\,dx =$ "

so $\quad \int_y^\infty (x-y)\, f(x)\,dx = c/\lambda$.

which has a unique solution y^*

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$$E[R(y^*)] = \frac{1}{1-F(y^*)}\left[\int_{y^*}^\infty (x - y^* + y^*)\, f(x)\,dx - \frac{c}{\lambda}\right]$$
$$= \frac{1}{1-F(y^*)}\cdot\Big[\underbrace{\int_{y^*}^\infty (x-y^*)\, f(x)\,dx}_{c/\lambda} + \underbrace{y^*\int_{y^*}^\infty f(x)\,dx}_{y^*[1-F(y^*)]} - \frac{c}{\lambda}\Big]$$
$$= y^*$$

optimal expected return = optimal critical value.

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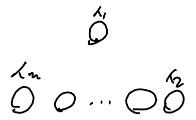
Example



- Particles at each node
- Moves according to P_{ij} at each event
- At time 0 number in state i, $i=1,\ldots,n \sim Po(\lambda_i)$

$N_j(i)$ is number initially at state i and state j at time n.

$N_j(i) \sim \text{Poisson}(\lambda_i P_{ij}^n)$

and $\sum_{j=1}^{n} N_j(i)$ $\text{Poisson}(\sum_{i=1}^{m} \lambda_i P_{ij}^n)$

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Example, coupon collector's problem

- m different coupons
- new coupon type j with probability p_j, $\sum_{j=1}^{m} p_j = 1$
- N number coupon needed for a full collection.

$E(N) = ?$

N_j number of coupons needed to get a coupon of type j

N_j geometric distributed, N_j

so $N = \max(N_j)$ by N_j's dependent.

Alternative: Coupons are collected at times following a Poisson process, rate $\lambda = 1$

$N_j(t)$, $j=1,\ldots,m$ associated counting process for m types

X_j first event process $N_j(t)$

$X = \max_{1\le j\le m}(X_j)$ time to a complete collection

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$P(X<t) = P(\max X_j < t) = P(\text{all } X_j < t)$

$= \prod_{j=1}^{m}(1-e^{-p_j t})$

and $E[X] = \int_0^\infty x f(x)dx = \int_0^\infty \underbrace{\int_0^\infty I[x>t]dt}_{x} f(x)dx$

$= \int_0^\infty \underbrace{\int_0^\infty I[x>t] f(x)dx}_{P(X>t)} dt$

$= \int_0^\infty [1-\prod_{j=1}^{m}(1-e^{-p_j t})]dt$.

Let $T_1, T_2, \ldots$ be interarrival times in process counting coupons.

$X = T_1 + \cdots + T_N$

N indenent of $T_1, T_2, \ldots$ $T \sim \exp(\lambda)$, $\lambda = 1$.

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$E[X|N] = N\cdot E(X) = N$

$E(X) = E[E[X|N]] = E(N)$

Let $I_i = \begin{cases} 1 & \text{only 1 coupon of type } i \text{ in collection} \\ 0 & \text{else} \end{cases}$

$E[\sum_{i=1}^{m} I_i] = \sum_{i=1}^{m} P(I_i = 1) = ?$

Define S_i second time a coupon type i appear

Then $I_i = 1 \iff X_j < S_i \ \forall j \ne i$

S_i is gamma$(2, p_i)$

$P(I_i = 1) = \int_0^\infty P(X_j < S_i \ \forall j\ne i \mid S_i = x) \cdot p_i^2 \cdot x \cdot \exp(-p_i x)dx$

$= \int_0^\infty \underbrace{P(X_j < x \ \forall j \ne i)} \cdot p_i^2 x \exp(-p_i x)dx$

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$= \int_0^\infty \prod_{j\ne i}(1-e^{-p_j x}) \cdot p_i^2 x \exp(-p_i x)dx$

$= \int_0^\infty x \cdot \prod_{j=1}^{m}(1-e^{-p_j x}) \cdot p_i^2 \cdot \frac{\exp(-p_i x)}{1-e^{-p_i x}} dx.$

$E(I_i)$

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Consider two independent Poisson processes rates λ_1 and λ_2

Let S_n^i be time of n'th event in i'th process

Then $P(S_1^1 < S_1^2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$

But $P(S_2^1 < S_1^2) = ?$ i.e. two events in process 1 before first event process 2.

Use memoryless property. The process start over again after first arrival of in process 1.

so therefore $P(S_2^1 < S_1^2) = \frac{\lambda_1}{\lambda_1+\lambda_2}\cdot\frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{\lambda_1^2}{(\lambda_1+\lambda_2)^2}$

n events of type 1 before m events of type 2 $\iff$ $n+m-1$ trials n or more successes

$$P(S_n^1 < S_m^2) = \sum_{k=n}^{n+m-1}\binom{n+m-1}{n}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n+m-1-k}$$

$$= \sum_{k=n}^{n+m-1}\binom{n+m-1}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n+m-1-k}$$

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