5.3.5 Conditional distribution of arrival times.

X Poisson process with rate λ.

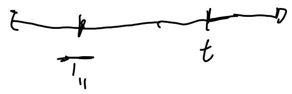
First: Distribution of T_1 given $N(t)=1$

i.e $P(T_1 < s \mid N(t)=1) =$

$$\frac{P(T_1<s,\ N(t)=1)}{P(N(t)=1)} = \frac{P(\text{one event in } [0,s) \text{ and no event in } [s,t])}{P(N(t)=1)}$$

$$\frac{\frac{(\lambda s)^1}{1!}e^{-\lambda s}\cdot\frac{[\lambda(t-s)]^0}{0!}\cdot e^{-\lambda(t-s)}}{\frac{(\lambda t)^1}{1!}\cdot e^{-\lambda t}} = \frac{s}{t} \quad s<t$$

which is the uniform distribution on $[0,t]$



mar. 11-11:13

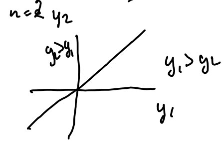
More generally, several events in $[0,t]$.

Order statistic: $Y_1,\dots,Y_n$ random variables

$Y_{(1)} < \dots < Y_{(n)}$ the ordered or sorted variables

If $Y_1,\dots,Y_n$ i.i.d $Y_i \sim f(y_i)$ then the order statistic has simultaneous density

$$n!\, f(y_1)\cdots f(y_n) \quad y_1<\dots<y_n$$



In general there are $n!$ regions where there is a 1-1 correspondence from $y_1,\dots,y_n$ to $y_{(1)}\dots y_{(n)}$

mar. 11-11:23

and the assignment of $Y_1,\dots,Y_n$ to any of these are equally likely since $Y_1,\dots,Y_n$ are i.i.d. The order statistic is the same for all these regions. i.e all $n!$

so therefore the p.d.f of $Y_{(1)}<\dots<Y_{(n)}$ is

$$n!\, f(y_1)\cdots f(y_n) \quad y_1<\dots<y_n$$

If Y_i, $i=1,\dots,n$ are independent and uniformly distributed over $[0,t]$ the p.d.f. of the order statistic is

$$f(y_1,\dots,y_n) = \begin{cases} \frac{n!}{t^n} & 0<y_1<\dots<y_n \\ 0 & \text{else} \end{cases}$$

mar. 11-11:31

Theorem 5.2. Given that $N(t)=n$, the n arrival times have the same distribution as the order statistic corresponding to n independent random variables which are uniformly distributed on $[0,t)$.

Proof. If $0<s_1<\dots<s_n<t$

$\{S_1=s_1,\dots,S_n=s_n, N(t)=n\} \iff \{T_1=s_1, T_2=s_2-s_1,\dots,T_n=s_n-s_{n-1}, T_{n+1}>t-s_n\}$

But $T_1,\dots,T_n$ are independent, $T_i \sim \text{Exp}(\lambda)$ (Proposition 5.4)

$$f(s_1,\dots,s_n \mid N(t)=n) = \frac{f(s_1,\dots,s_n,n)}{P(N(t)=n)} =$$

$$\frac{\lambda e^{-\lambda s_1}\cdot\lambda e^{-\lambda(s_2-s_1)}\cdots\lambda e^{-\lambda(s_n-s_{n-1})}e^{-\lambda(t-s_n)}}{\frac{(\lambda t)^n}{n!}e^{-\lambda t}}$$

$$= \frac{n!}{t^n} \quad 0<s_1<\dots<s_n<t$$

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mar. 11-11:37

Remark 5.3.5 Theorem 5.2 is often expressed as $S_1,\dots,S_n$ is considered as unordered random variables which are independent and uniformly distributed on $[0,t)$.

Example insurance

- Claims arrive according to a Poisson process with rate λ. S_n is the waiting time to n'th claim
- Claims amount random variable C_i with c.d.f G, $E(C)=\mu$.
- C_i, $i=1,2,\dots$ are independent of the claims arrival times

Total discounted cost at time t

$$D(t) = \sum_{i=1}^{N(t)} C_i\, e^{-\alpha S_i}$$

$E[D(t)]$?

mar. 11-11:49

Condition on $N(t)=n$

$$E[D(t)] = \sum_{n=0}^{\infty} E[D(t) \mid N(t)=n]\cdot\frac{(\lambda t)^n}{n!}e^{-\lambda t}$$

Conditional on $N(t)$ the claim arrivals are distributed as $(U_{(1)},\dots,U_{(n)})$ $U_i \sim [0,t]$

Hence $E[D(t) \mid N(t)=n] = E\left[\sum_{i=1}^{n} C_i e^{-\alpha S_i}\right]$

$$= \sum_{i=1}^{n} E(C_i)\cdot E[e^{-\alpha S_i}]$$

$$= \mu\cdot\sum_{i=1}^{n} E[e^{-\alpha U_{(i)}}]$$

But $\sum_{i=1}^{n} e^{-\alpha U_{(i)}} = \sum_{i=1}^{n} e^{-\alpha U_i}$

so $E[D(t) \mid N(t)=n] =$

$$n\cdot\mu\cdot E[e^{-\alpha U}] =$$

mar. 11-11:55

$$= n \cdot \frac{\mu}{t} \cdot \int_0^t e^{-\alpha x} dx$$

$$= \frac{n \cdot \mu}{\alpha t} \cdot [1 - e^{-\alpha t}]$$

so $E[D(t) \mid N(t) = n] = \underbrace{N(t)}_{\lambda t} \cdot \frac{\mu}{\alpha t} \cdot [1 - e^{-\alpha t}]$

and $E[D(t)] = \frac{t \cdot \lambda \cdot \mu}{\alpha t} [1 - e^{-\alpha t}]$

$$= \frac{\lambda \cdot \mu}{\alpha} [1 - e^{-\alpha t}].$$

mar. 11-12:00