Thm 5.2 $S_1, \dots, S_n \mid N(t)=n \sim U_{(1)} < \dots < U_{(n)}$, U_i i.i.d $U[0,t]$

Rember in Prop 5.2 the events were classified as type I or type II with probabilities p and $1-p$ respectively.

What happens if classification also involves time of event?

Assume that an event occurring at time y be classified as a "type i" event independently of previous events with a probabilit $P_i(y)$, $i=1,\dots,k$ $\quad \sum_{i=1}^{k} P_i(y) = 1$

Proposition 5.3 If $N_i(t)$ represents the number of type i events occurring by time t, then $N_i(t)$, $i=1,\dots,k$ are independent Poisson random variables with means

$$E[N_i(t)] = \lambda \int_0^t P_i(s)\,ds$$

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Proof: Let $n_1, \dots, n_k$ be integers $\sum_{i=1}^{k} n_i = n$

$$P(N_1(t)=n_1, \dots, N_k=n_k) = P(N_1(t)=n_1, \dots, N_k(t), \sum_{i=1}^{k} N_i(t)=n)$$

$$= P(N_1(t)=n_1, \dots, N_k(t)=n_k \mid N(t)=n) \cdot P(N(t)=\sum n_i = n)$$

Consider an event i $[0,t]$. If it occurred at time s it would be a type i event with probability $P_i(s)$. Since the time of event is unif in $[0,t]$, (Theorem 5.2) the probability that the event is type i is $P_i = \frac{1}{t}\int_0^t P_i(s)\,ds$ independent of other events.

Hence, the joint conditional distribution given the number of events is multinomial

$$P(N_1(t)=n_1, \dots N_k(t)=n_k \mid N(t)=\sum_{i=1}^{k} n_i)$$

$$= \frac{\sum_{i=1}^{k} n_i}{n! \cdots n_k!} \cdot P_1^{n_1} \cdots P_k^{n_k} \qquad \sum_{i=1}^{k} P_i = 1$$

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Hence $P(N_1(t)=n_1, \dots, N_k(t)=n_k) =$

$$\frac{\cancel{(\sum n_i)!}}{n_1! \cdots n_k!} P_1^{n_1} \cdots P_k^{n_k} \frac{(\lambda t)^{\sum n_i}}{\cancel{(\sum n_i)!}} e^{-\lambda t}$$

$$= \prod_{i=1}^{k} \frac{(\lambda t P_i)^{n_i}}{n_i!} e^{-\lambda t P_i} \quad \text{since } \sum P_i = 1$$

But $\lambda t P_i = \lambda \cdot \int_0^t P_i(s)\,ds$

Example $M/G/\infty$ queue

- M: customers arrive according to a Poisson process rate λ
- G: service time has a general distribution, with c.d.f G and an independent
- ∞: infintly many serves.

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Let $X(t)$ number of clients completed service by t

$Y(t)$ — " — not completed — " —

Distribution of $X(t)$ and $Y(t)$?

type I: customer completed service by t

type II — " — not completed — " —

Customer arrives at $s \le t$ $\begin{cases} \text{type I with probability } G(t-s) \\ \text{type II} \quad \text{—"—} \quad 1-G(t-s) \end{cases}$

Using proposition 5.3

$$X(t) \sim Po\left(\lambda \int_0^t G(y)\,dy\right)$$

since $\lambda \cdot \int_0^t G(t-s)\,ds = -\int_t^0 G(y)\,dy = \int_0^t G(y)\,dy$

and $Y(t) \sim Po\left(\lambda \cdot \int_0^t (1-G(y))\,dy\right)$

Also from prop 5.3 $X(t)$ and $Y(t)$ are independents

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Joint distribution of $Y(t)$ and $Y(t+s)$?

type 1: arries $[0,t]$ completes in $[t, t+s)$

type 2: arrives $[0,t]$ comples $[t+s, \infty)$

type 3: arrives $[t, t+s)$ completes $[t+s, \infty)$

type 4 others.

Now, $P_i(y)$ is the probability for a customer arriving at y being a type i customer

Then $P_1(y) = \begin{cases} G(t+s-y) - G(t-y) & \text{if } y<t \\ 0 & \text{else} \end{cases}$

because the service time is between $t-y$ and $t+s-y$

$P_2(y) = \begin{cases} 1 - G(t+s-y) & \text{if } y<t \\ 0 & \text{else} \end{cases}$

$P_3(y) = \begin{cases} 1 - G(t+s-y) & t<y<t+s \\ 0 & \end{cases}$

$P_4(y) = 1 - P_1(y) - P_2(y) - P_3(y)$.

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From proposition 5.3 if $N_i = N_i(t)$

$$E(N_i) = \lambda \cdot \int_0^{t+s} P_i(y)\,dy$$

$Y(t) = N_1 + N_2$ customers completing later than t

$Y(t+s) = N_2 + N_3$ — " — than $t+s$

Then $Cov(Y(t), Y(t+s)) = Cov(N_2, N_2) = Var(N_2)$ since the N_i are independent

$$= \lambda \cdot \int_0^t [1 - G(t+s-y)]\,dy$$

$$= \lambda \cdot \int_0^t [1 - G(y+s)]\,dy$$

since N_2 are is Poisson.

Similarly $P(Y(t)=i, Y(t+s)=j)$

$$= P(N_1 + N_2 = i, N_2 + N_3 = j)$$

$$= \sum_{l=0}^{\min(i,j)} P(N_2 = l, N_1 = i-l, N_3 = j-l)$$

$$= \sum_{l=0}^{\min(i,j)} P(N_2 = l) P(N_1 = i-l) P(N_3 = j-l).$$

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5.4 Generalizations of the Poisson process.

5.4.1 Non-homogenous / non-stationary Poisson process.

In the non-homogenous Poisson process the arrival rate is allowed to be a function of t.

Definition: The counting process in a non-homogenous Poisson process with intensity function $\lambda(t)$, $t \geq 0$ if

(i) $N(0) = 0$

(ii) $\{N(t), t \geq 0\}$ has independent increments

(iii) $P(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$

(iv) $P(N(t+h) - N(t) \geq 2) = o(h)$

The function $m(t) = \int_0^t \lambda(y)\,dy$ is the mean value function of the non-homogenous Poisson process

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Theorem 5.3. If $\{N(t), t \geq 0\}$ is a non-homogenous/non-stationary Poisson process with intensity function $\lambda(t)$, $t \geq 0$, then $N(t+s) - N(s)$ is a Poisson random variable with mean/expectation function

$$m(t+s) - m(t) = \int_s^t \lambda(y)\,dy.$$

Proof. Generalization of Theorem 5.1 where $\lambda(t) = \lambda$. Let $g(t) = E[e^{-uN(t)}]$ be the Laplace transform. Using independent increment requirement (ii)

$$g(t+h) = E e^{-u[N(t+h) - N(t)] - uN(t)}$$

$$= g(t) \cdot E e^{-N_t(h)}$$

where $N_t(h) = N(t+h) - N(t)$

Using (iii) & (iv)

$$g(t+h) = g(t) \cdot [\underbrace{1 - \lambda(t)h + o(h)}_{\text{"}N_t(h) = 0\text{"}}] + \underbrace{e^{-u}\lambda(t)h}_{N_t(h)=1} + \underbrace{o(h)}_{N_t(h)\geq 2}$$

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such $\frac{g(t+h) - g(t)}{h} = g(t)\lambda(t) \cdot [e^{-u} - 1]h + o(h)$.

Let $h \to 0$ $g'(t) = g(t)\lambda(t)(e^{-u} - 1)$

$$\frac{\partial}{\partial t} \log g'(t) = \lambda(t) \cdot (e^{-u} - 1)$$

Integrating

$$\log(g(t)) - \underbrace{\log(g(0))}_{0} = (e^{-u} - 1) \cdot \underbrace{\int_0^t \lambda(t)\,dt}_{= m(t)}$$

$$g(t) = \exp(m(t)(e^{-u} - 1))$$

so by uniqueness for the Laplace transform

$$N(t) \sim \text{Poisson}(m(t))$$

To conclude that $N(s+t) - N(s) = N_s(t)$ also is Poisson, note that $N_s(t)$ is non-stationary Poisson with intensity function $\lambda_s(t) = \lambda(s+t)$ $t \geq 0$

Hence $N_s(t) \sim \text{Poisson}\left(\int_0^t \lambda_s(y)\,dy\right)$

But $\int_0^t \lambda_s(y) = \int_0^t \lambda(s+y)\,dy = \int_s^{s+t} \lambda(u)\,du$

$$= m(t+s) - m(t)$$

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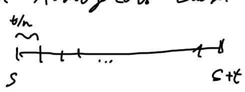
Remark 5.4.1 Use of the Poisson approximation to the binomial distribution is also useful in the non-homogenous case.



Number of events is approximately equal to the number of intervals where event occurs. But this is a sum of 0,1 variables which are independent, and

$P(\text{event in } i\text{th interval}) = \lambda(s + i\frac{t}{n}) \cdot \frac{t}{n} + o(\frac{t}{n})$

and the expectation of sum is

$$\sum_{i=1}^{n} \lambda(s + i\tfrac{t}{n}) \cdot \tfrac{t}{n} + n \cdot o(\tfrac{t}{n}) \to \int_s^{s+t} \lambda(y)\,dy$$

and the distribution of the sum is approximately Poisson.

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Remark 5.4.2 If events occur according to a homogenous process the event is counted/sampled with probability $p(t)$ independently of what occurred prior to t.

Then the number of counted/sampled events is a non-homogenous Poisson process with mean function $\int_0^t \lambda(y)\,dy$.

This generalizes to the situation where there are two types of event and the probabilities of an event being of type I or type II are $P_1(s)$ or $P_2(s) = 1 - P_1(s)$ independent of what happened prior to t

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Then $N_1(t)$ and $N_2(t)$ are independent Poisson with $E[N_i(t)] = \int_0^t P_i(s)\,ds$ when $N_i(t)$ are the number of type i events occured at time t.

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