5.4.2 Compound Poisson Processes

$\{X(t), t \geq 0\}$ is a compound Poisson process if it can represented as a cumulative series

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0$$

and ~~independent~~ of $\{N(t), t \geq 0\}$ process rate λ. $Y_1, Y_2, \ldots$ are i.i.d variables

Example (i) $Y_i = 1$, $\Rightarrow X(t)$ Poisson process

(ii) Customers leaving a supermarket

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according to a Poisson process

Y_i amount spent by customer i

Then $X(t)$ is the total amount spent if the assumption of independence is true.

(iii) claim sizes Y
claim process Poisson
reasonable assumption in insurance

$E[X(t)] = ?$

$$E[X(t)] = E\left[\sum_{i=1}^{N(t)} Y_i\right]$$

But $E\left[\sum_{i=1}^{N(t)} Y_i \mid N(t) = n\right] = E\left[\sum_{i=1}^{n} Y_i \mid N(t) = n\right]$
$= n \cdot E(Y_i)$

so $E\left[\sum_{i=1}^{N(t)} Y_i \mid N(t)\right] = N(t) \cdot E(Y_1)$

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and $E\left(\sum_{i=1}^{N(t)} Y_i\right) = E[N(t)] \cdot E(Y_1) = \lambda t \cdot E(Y_1)$.

$Var(X(t)) = ?$

- $Var\left(\sum_{i=1}^{N(t)} Y_i \mid N(t) = n\right) = Var\left(\sum_{i=1}^{n} Y_i \mid N(t) = n\right)$
$= n \cdot Var(Y_1)$
- Have shown $E\left[\sum_{i=1}^{N(t)} Y_i \mid N(t)\right] = N(t) \cdot E(Y_1)$

$Var\left(E\left[\sum_{i=1}^{N(t)} Y_i \mid N(t)\right]\right) = [E(Y_1)]^2 \cdot Var(N(t)) = \lambda t \cdot E(Y_1)^2$

Example. Watching birds of a certain specie
2 cluster of birds per day
Y_i size of cluster i, $P(Y=1) = 0.1$
$P(Y=2) = 0.4$
$P(Y=3) = 0.4$
$P(Y=4) = 0.1$

$E(Y) = \frac{1}{10} + \frac{4}{10} \cdot 2 + \frac{4}{10} \cdot 3 + \frac{1}{10} \cdot 4 = \frac{25}{10} = 2.5$

$E(Y^2) = \frac{1}{10} + \frac{4}{10} 2^2 + \frac{4}{10} 3^2 + \frac{1}{10} 4^2 = \frac{69}{10} = 6.9$

So the expected birds seen in a week is

$E[X(t)] = 7 \times 2 \times 2.5 = 35$

$Var(X(t)) = 7 \cdot 2 \cdot \frac{69}{10} =$

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Example Busy period of an M/G/1 queue

Arrivals of customers : Poisson rate λ

Service time : General distribution, c.d.f G

number of servers : 1

Customers are served in order of arrival by server, and join the queue if he/she is busy. Hence idle and busy periods of server alternates. By memory less property of Poisson arrivals the busy periods are i.i.d.

$E(B) = ?$ $Var(B) = ?$

Let S be the service time of the first customer in a busy period. Then $N(S)$ is the number of customers arriving during the first service, i.e in $[0, S]$

If $N(S) = 0$ the busy period ends

If $N(S) = 1$ one customer has arrived

Arrivals after S will be Poisson, so time from S til the system is empty will have distribution as busy period,

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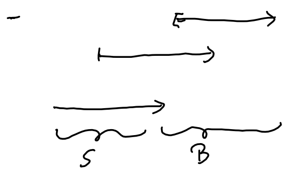

$B \sim S + B$

$= S + \sum_{i=1}^{N(S)} B_i$

($X \sim Y$ mean that X and Y have same distribution)

In general, if $N(S) = n$ there have arrived n customers during the first service. Let the arrivals be $C_1, \ldots, C_n$

Suppose they are served as follows

C_1 is served first, but any new customer arriving during C_1's service time is served before customer C_2. Similarly

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C_2 is not served before customers arriving during C_1's service is completed, and so

Thus time between serving C_i an C_{i+1} $i = 1, \ldots, n-1$

are i.i.d distributed as B

Hence $B \sim S + \sum_{i=1}^{N(S)} B_i$

so $E[B \mid S] = S + E\left[\sum_{i=1}^{N(S)} B_i \mid S\right]$

$Var[B \mid S] = Var\left(\sum_{i=1}^{N(S)} B_i \mid S\right)$

Remember that

$E[X(t)] = \lambda t \, E(Y_1)$

$Var(X(t)) = \lambda t \, E(Y_1^2)$

substituting S for t

(i) $E[B \mid S] = S + \lambda S \, E(B) = S[1 + \lambda \cdot E(B)]$

(ii) $Var(B \mid S) = \lambda \cdot S \cdot E(B^2)$

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Rearranging $E(B) = [1+\lambda E(B)]E(S)$

$$E(B) = \frac{E(S)}{1-\lambda \cdot E(S)}$$

if $\lambda \cdot E(S) < 1$.

Similarly

$$Var(B) = Var(E[B|S]) + E[Var(B|S)]$$

$$= [1+\lambda E(B)]^2 \cdot var(S) + \lambda E(S) E(B^2)$$

"ii" "i"

$Var(B) + [E(B)]^2$

Then $$Var(B) = \frac{Var(S)[1+\lambda E(B)]^2 + \lambda E(S) \cdot E(B)^2}{1-\lambda E(S)}$$

Inserting $E(B) = \frac{E(S)}{1-\lambda E(S)}$

and $$1+\lambda E(B) = \frac{1-\lambda E(S)+\lambda E(S)}{1-\lambda E(S)} = \frac{1}{1-\lambda E(S)}$$

$$Var(B) = \frac{Va(S) + \lambda E(S)^3}{[1-\lambda E(S)]^3}$$

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Consider the case when the distribution of $Y_1 ..$ is discrete

$$P(Y_i = \alpha_j) = p_j, \quad \sum p_j = 1$$

In a compound Poisson process a random amount is added to the cumulative sum $X(t)$ each time an event occurs.

Let $N_j(t)$ be the counting process the times α_j has been added. Then we know that $N_j(t)$ $j = 1, ..$ are independent random variables with means $E[N_j(t)] = \lambda \cdot p_j \cdot t$ (proposition 5.2)

It follows that $X(t) = \sum_{j=1} \alpha_j N_j(t)$

As before $E[X(t)] = \lambda \cdot t \cdot E(Y_1)$

$Var(X(t)) = \lambda \cdot t \; E(Y_1^2)$

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so

$$E[X(t)] = E[\sum \alpha_j N_j(t)] = \sum_j \alpha_j \lambda t p_j = \lambda \cdot t \, E(Y_1)$$

and

$$Var(X(t)) = Va(\sum \alpha_j N_j(t)) = \sum \alpha_j^2 \lambda \cdot t \cdot p_j = \lambda \cdot t \, E(Y^2)$$

A useful consequence of

$$X(t) = \sum_{j=1}^{\infty} \alpha_j N_j(t)$$

is that $X(t)$ is a sum of independent Poisson$(\alpha_j p_j \lambda t)$ variables. Then for large t

$$\frac{X(t) - E(X(t))}{\sqrt{Va(X(t))}}$$ can be approximated by a standard normally distributed variable.

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If $X \sim Po(n)$ $n \to \infty$

$\frac{X-n}{\sqrt{n}}$ approximately $N(0,1)$

since $X = \sum_{i=1}^{n} X_i$, $X_i \sim Po(1)$

$$\frac{X-n}{\sqrt{n}} = \frac{\sum X_i - n}{\sqrt{n}} = \frac{n(\bar{X}-1)}{\sqrt{n}} = \frac{\sqrt{n}(\bar{X}-1)}{1}$$

which is approximately $N(0,1)$ by the CLT, t

This can be used to approximate

$$P(X(t) > k) = P\left(\underbrace{\frac{X(t)-E(X(t))}{\sqrt{Va(X(t))}}}_{\approx N(0,1)} > \frac{k - E(X(t))}{\sqrt{Var(X(t))}}\right)$$

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Another useful result for compound Poisson processes.

If $X(t) = \sum_{i=1}^{N_x(t)} Z_{xi}$

$Y(t) = \sum_{i=1}^{N_y(t)} Z_{yi}$

are independent compound Poisson processes, then $N_x(t) + N_y(t)$ is a Poisson process rate $\lambda_x + \lambda_y$

The events in $N_x(t) + N_y(t)$ will be from $X(t)$ with probability $\frac{\lambda_x}{\lambda_x + \lambda_y}$

Therefore $X(t) + Y(t)$ will be a compound Poisson process when $N_x(t) + N_y(t)$ have rate $\lambda_x + \lambda_y$ and the i.i.d variables Y_{xy} will have a mixture distribution with c.d.f $\frac{\lambda_x}{\lambda_x+\lambda_y} F_x + \frac{\lambda_y}{\lambda_x+\lambda_y} \cdot F_y$.

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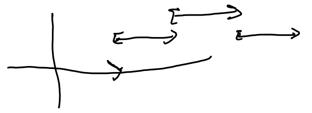
Chapter 6. Continuous time Markov processes

Here combine results from chapter 4 and chapter 5.

There are events occurring in sequence

- The time between events are not necessarily i.i.d. They are exponential by expectation can vary
- The process alternates between a set of states. Here a transition matrix is used to describe the moves.



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