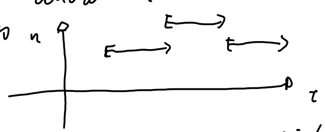
Chapter 6. Continuous time Markov chains

Here we consider a sequence of events but there are two modifications compared to earlier.

- The times between events are not necessarily i.i.d. The distributions are still exponential and independent but the mean times / expectations may vary.
- The process alternates between a set of states. If, when an event occurs, the process have been in state i, it moves to state j according to a transition matrix P

Typical example. Birth and death process which allows transitions to adjacent states



Poisson process is a special case

Definition: A process $\{X_t, t\geq 0\}$ is a continuous-time Markov chain if for all $s,t\geq 0$ and non-negative integers $i, j, x(u)$ $0\leq u<s$

$$P(X(t+s)=j \mid X(s)=i,\ X(u)=x(u),\ 0\leq u<s)$$
$$=P(X(t+s)=j \mid X(s)=i)$$

- So the distribution of the future depends only the present and is independent of the past
- In addition $P(X(t+s)=j \mid X(s)=i)$ will be assumed to depend only on s, so the process has stationary increments

The process is defined by

- each time it enters a state i it spends an exponentially distributed time, $T_i \sim v_i \exp(-v_i x)$, there before transition to another.

Remember that the exponential distribution has a memoryless property.

$$P(T>s+t \mid T>s)=P(T>t)$$

- When the process leaves state i it enters the next state j according to a transition matrix P

NB. The moves between states are independent of how long time the process has spent in the previous and present state.

Example: Two machines is in series, only admission to system if both are free

Machine 1 Machine 2

$$\xrightarrow{\lambda} \boxed{\mu_1} \longrightarrow \boxed{\mu_2}$$

Summary of system:

States	
0	system is empty
1	machine one working
2	—"— two —"—

Here $v_0=\lambda,\ v_1=\mu_1,\ v_2=\mu_2$

$$P=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

6.3. Birth and death processes.

These processes are continuous-time Markov chains with states $\{0,1,\dots\}$

$v_0=\lambda_0$

$v_i=\lambda_i+\mu_i \quad i>0$

this describes the time spent in particular states

transition described by

$P_{01}=1$

$P_{i,i+1}=\dfrac{\lambda_i}{\lambda_i+\mu_i} \quad i>0$

$P_{i,i-1}=\dfrac{\mu_i}{\lambda_i+\mu_i} \quad i>0$

$$P=\begin{pmatrix} 0 & 1 & 0 & \cdots & & \\ \frac{\mu_1}{\lambda_1+\mu_1} & 0 & \frac{\lambda_1}{\lambda_1+\mu_1} & 0 & \cdots & \\ 0 & \frac{\mu_2}{\lambda_2+\mu_2} & 0 & \frac{\lambda_2}{\lambda_2+\mu_2} & 0 & \cdots \\ & & \ddots & & & \end{pmatrix}$$

- Transitions upward are "birth" and downward "death" are possible. λ_i arrival/birth rate, μ_i departure/death rate
- If the process is in state $i\geq 1$, the time to the next event is exponentially distributed $\exp(-x\underbrace{(\lambda_i+\mu_i)}_{v_i})$ and the probability that the event is birth is $\frac{\lambda_i}{\lambda_i+\mu_i}$

—— " —— death is $\frac{\mu_i}{\lambda_i+\mu_i}$

- If the process is in state 0 the next event will be a birth and the waiting time is $v_0\exp(-x v_0)$ distributed.

Example: If $\mu_n=0 \quad n\geq 0$

$\lambda_n=\lambda \quad n\geq 0$

then no death and the process always jumps upward, and time between events are i.i.d $\lambda\exp(-\lambda x)$.

This characterizes a Poisson process.

Example. Birth process with linear birth rate. $v_n=n\lambda \quad n\geq 0$

$\mu_n=0 \quad$ all n.