Birth and death processes:

$\{X(t), t \geq 0\}$ individuals at time t

$\{\lambda_i; i \geq 0\}$ $\{\mu_i; i \geq 0\}$

T_i times between events

$T_n = \min(U_n, V_n)$ $n \geq 1$

$U_n \sim \mu_n \exp(-\mu_n x)$, $V_n \sim \lambda_n \exp(-\lambda_n x)$

U_n and V_n are independent

$\Rightarrow T_n \sim (\mu_n + \lambda_n)\exp(-(\lambda_n + \mu_n)x)$

Also $T_0 \sim \nu_0 \exp(-\nu_0 x)$.

Example Birth process with linear birth rate

$\nu_n = n\lambda$ $n \geq 1$, $X(0) = i > 0$

$\mu_n = 0$ all

This can be a model for a population where no individual die, each one acts independently and gives birth with exponential rate λ. The total rate for a new birth is then $n\lambda$. Yule process

Example: A linear growth model with immigration

Then $\mu_n = n\mu$ $n > 0$

$\lambda_n = n\lambda + \theta$ $n \geq 0$

The model can be used for biological reproduction and population growth

Each individual gives birth with rate λ, so $n\lambda$

" " dies " " μ, so $n \cdot \mu$

In addition there is a constant rate of increase with rate θ, due to an external source.

Let $X(t)$ be population size at time t and $X(0) = i$

$E[X(t)] = M(t)$?

Compare times $t+h$ and h.

$$X(t+h) = \begin{cases} X(t)+1 & \text{with probability } [\theta + X(t)\cdot\lambda]h + o(h) \\ X(t)-1 & \text{" "} \quad X(t)\cdot\mu h + o(h) \\ X(t) & \text{" "} \quad 1 - [\theta + X(t)\mu + X(t)\lambda]h + o(h) \end{cases}$$

so $E[X(t+h) \mid X(t)] = (X(t)+1)(\theta + X(t)\lambda)h$
$+ X(t)[1 - (\theta + X(t)\lambda + X(t)\mu)]h$
$+ (X(t)-1)(X(t)\mu h) + o(h)$

$= X(t)^2[\lambda - \lambda - \mu + \mu]$
$+ X(t)[1 + \theta - \theta + \lambda - \mu]h + \theta h + o(h)$

$= X(t) + [\theta + X(t)(\lambda - \mu)]h + o(h)$.

$= E[X(t+h) \mid X(t)]$. By iterated expectation

$M(t+h) = M(t) + (\lambda - \mu)M(t) + \theta h + o(h)$

$\Rightarrow \frac{M(t+h) - M(t)}{h} = \underbrace{M(t)(\lambda-\mu) + \theta}_{h(t)}$

$\Rightarrow M'(t) = M(t)(\lambda - \mu) + \theta$

Then $h'(t) = M'(t)(\lambda - \mu) \Rightarrow M'(t) = h'(t)(\lambda - \mu)$

so $\frac{h'(t)}{h(t)} = \lambda - \mu$.

$\frac{\partial}{\partial t}\log(h(t))$ when $\mu \neq \lambda$

$\log(h(t)) = (\lambda - \mu)t + k$

$h(t) = K \cdot \exp((\lambda - \mu)t)$

Inserting $M(t)(\lambda - \mu) + \theta = K \cdot \exp((\lambda - \mu)t)$

Now $X(0) = M(0) = i$

$\theta + i(\lambda - \mu) = K$

$\Rightarrow \mu(t)(\lambda - \mu) + \theta = [\theta + i(\lambda - \mu)]\exp((\lambda - \mu)t)$

$\rightarrow M(t) = E(X(t)) = \frac{\theta}{\lambda - \mu}[\exp((\lambda-\mu)t) - 1] + i\exp((\lambda - \mu)t)$

If $\lambda = \mu$ $M'(t) = \theta$

$M(t) = \theta t + K$, $K = M(0) = i$

so $M(t) = \theta t + i$, only immigration

Example $M/M/1$ queue

arrivals: Poisson process rate λ

service times: i.i.d exponential $\mu\exp(-\mu x)$

queuing discipline: If server free when new customer arrives, customer served at once, else join the queue. When service finished customer leaves. "FIFO"

Let $X(t)$ number in system

$X(t)$ birth and death process

$\mu_n = \mu$ $n \geq 1$

$\lambda_n = \lambda$ $n \geq 0$

Example: Multiserver exponential queuing system $M/M/s$.

s servers and entering customers wait in a line and are served by the first free server.

$X(t)$: number in system

$X(t)$ birth and death process

Here $\mu_n = \begin{cases} n\mu & 1 \leq n \leq s \\ s\mu & n > s \end{cases}$

$\lambda_n = \lambda$ $n \geq 0$

Consider a birth and death process with rates $\{\mu_n, n \geq 0\}$ $\mu_0 = 0$, $\{\lambda_n, n \geq 0\}$

Let T_i be the time from entering state i until entering state $i+1$

$E[T_0] = 1/\lambda_0$ since $T_0 \sim \lambda_0\exp(-\lambda_0 x)$

But $E[T_i]$, $i \geq 1$?

Define $I_i = \begin{cases} 1 & \text{if first transition is } i \to i+1 \\ 0 & \text{" " } i \to i-1 \end{cases}$

Then $E[T_i \mid I_i = 1] = \frac{1}{\lambda_i + \mu_i}$

since the first transition is $i \to i+1$ and T_i is therefore the time until first event which is exponential with mean $\frac{1}{\lambda_i + \mu_i}$

and time until transition is independent of whether the new state is $i+1$ or $i-1$.

But if the first event is $i \to i-1$ the population size decreases and an additional time is needed to reach state i again and then state $i+1$.

Hence $E[T_i \mid I_i = 0] = \frac{1}{\lambda_i + \mu_i} + E[T_{i-1}] + E[T_i]$

The probability that the first transition is a birth is $\frac{\lambda_i}{\lambda_i + \mu_i}$ so the unconditional expectation is

$$E[T_i] = \frac{1}{\lambda_i + \mu_i} \cdot \frac{\lambda_i}{\lambda_i + \mu_i} + \left[\frac{1}{\lambda_i + \mu_i} + E[T_{i-1}] + E[T_i]\right] \cdot \frac{\mu_i}{\lambda_i + \mu_i}$$

$$\Rightarrow E[T_i]\left[1 - \frac{\mu_i}{\lambda_i + \mu_i}\right] = \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i} E[T_{i-1}]$$

$\Rightarrow \quad \lambda_i E(T_i) = 1 + \mu_i E[T_{i-1}]$

$$E(T_i) = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E[T_{i-1}], \quad i = 1, 2, \ldots$$

so we get a recursion starting with

$E[T_0] = \frac{1}{\lambda_0}$.

Generalizing a bit: The expected time to go from i to $j > i$ is

$$E[T_i] + E[T_{i+1}] + \ldots E[T_{j-1}].$$

Example: $\lambda_i = \lambda$, $\mu_i = \mu$

inserting $E(T_i) = \frac{1}{\lambda} + \frac{\mu}{\lambda} E[T_{i-1}] = \frac{1}{\lambda}[1 + \mu E[T_{i-1}]]$

$E[T_0] = 1/\lambda$

$E(T_1) = \frac{1}{\lambda}[1 + (\frac{\mu}{\lambda})]$

$E[T_2] = \frac{1}{\lambda}[1 + (\frac{\mu}{\lambda})(1 + \frac{\mu}{\lambda})] = \frac{1}{\lambda}[1 + \frac{\mu}{\lambda} + (\frac{\mu}{\lambda})^2]$

continuing

$$E[T_i] = \frac{1}{\lambda}[1 + (\frac{\mu}{\lambda}) + (\frac{\mu}{\lambda})^2 + \cdots + (\frac{\mu}{\lambda})^i]$$

so $E(T_i) = \begin{cases} \frac{1}{\lambda} \cdot \frac{1 - (\frac{\mu}{\lambda})^{i}}{1 - \frac{\mu}{\lambda}} = \frac{1 - (\frac{\mu}{\lambda})^{i+1}}{\lambda - \mu} & \text{when } \mu \neq \lambda \\ \frac{i+1}{\lambda} & \mu = \lambda \end{cases}$

The expected time to reach state j starting at $k < j$ is

$$\sum_{i=k}^{j-1} E[T_i] = \frac{j-k}{\lambda - \mu} - \frac{1}{\lambda - \mu} \cdot \sum_{i=k}^{j-1} \left(\frac{\mu}{\lambda}\right)^{i+1}$$

?

$$= \frac{j-k}{\lambda - \mu} - \frac{1}{\lambda - \mu} \cdot \left(\frac{\mu}{\lambda}\right)^{k+1} \cdot \frac{1 - (\frac{\mu}{\lambda})^{j-k}}{1 - \frac{\mu}{\lambda}} \quad \text{when } \mu \neq \lambda$$

For $\lambda = \mu$:

$$\sum_{i=k}^{j-1} \frac{i+1}{\lambda} = \sum_{i=k+1}^{j} \frac{i}{\lambda} = \frac{j(j+1)}{2\lambda} - \frac{k(k+1)}{2\lambda}$$

What about $Var(T_i)$ for general birth and death processes.

Note that

$$E[T_i \mid I_i] = \frac{1}{\lambda_i + \mu_i} + (1 - I)[E[T_{i-1}] + E[T_i]].$$

so $Var[E(T_i \mid I_i)] = (E[T_{i-1}] + E[T_i])^2 \cdot Var(I)$ (*)

$$= (E(T_{i-1}) + E(T_i))^2 \cdot \frac{\lambda_i}{\lambda_i + \mu_i} \cdot \frac{\mu_i}{\lambda_i + \mu_i}$$

$E[Var(T_i \mid I_i)]$?

Let X_i be time until transition from i

$Var(T_i \mid I_i = 1) = Var(X_i \mid I_i = 1)$

$= Var(X_i)$

$= \frac{1}{(\lambda_i + \mu_i)^2}$ since $X_i \sim (\lambda_i + \mu_i) \exp(-(\lambda_i + \mu_i)x)$

similarly

$Var(T_i \mid I_i = 0) = Var(X_i + \text{time back to } i + \text{time } i+1 \text{ to reach})$

so the $Var(T_i \mid I_i) = Var(X_i) + (1 - I)[Var(T_{i-1}) + Var(T_i)]$

and $E[Var(T_i \mid I_i)] = \frac{1}{(\lambda_i + \mu_i)^2} + \frac{\mu_i}{\lambda_i + \mu_i}[Var(T_{i-1}) + Var(T_i)]$ (**)

so adding (*) and (**)

$$Var(T_i) = \frac{1}{(\lambda_i + \mu_i)^2} + \frac{\mu_i}{\mu_i + \lambda_i}[Var(T_{i-1}) + Var(T_i)] + \frac{\mu_i \lambda_i}{(\mu_i + \lambda_i)^2}[E(T_{i-1}) + E(T_i)]$$

simplifying

$$Var(T_i) = \frac{1}{\lambda_i(\mu_i + \lambda_i)} + \frac{\mu_i}{\lambda_i} Var(T_{i-1}) + \frac{\mu_i \lambda_i}{\mu_i + \lambda_i}[E[T_i] + E[T_{i-1}]]^2$$

so using the recursion for $E(T_i)$ one get an recursion for $Var(T_i)$ starting with $Var(T_0) = \frac{1}{\lambda_0^2}$.

- Finally: $Var(\text{time to go from } k < j) = \sum_{i=k}^{j-1} Var(T_i)$.