6.4 Transition probability function $P_{ij}(t)$

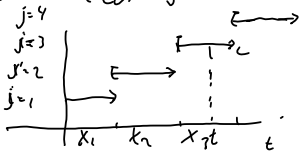
Let $P_{ij}(t) = P(X(s+t)=j \mid X(s)=i)$
$= P(X(t)=j \mid X(0)=i)$

be the transition probabilities of a continuous time Markov chain with stationary transition probabilities

Consider a pure birth process $\lambda_i \neq \lambda_j$ $i \neq j$ which is presently in state i, so $X(0)=i$

Let X_i be the time spent in state i before moving to $i+1$
X_{i+1} — " — $i+1$ — " — $i+2$
etc

Then $X(t) < j \iff X_1 + \dots + X_{j-1} > t$



$X_1 \dots X_j$ are independent $\sim \lambda_i \exp(-\lambda_i x)$ $i=1,\dots,j-1$

Recall (6.9) p 295 in Ross
hypoexponential distribution:
If $Y_1 \dots Y_n$ are independent $Y_i \sim \lambda_i \exp(-\lambda_i x)$ $\lambda_i \neq \lambda_j$

Then $P(Y_1 + \dots + Y_n > t) = \sum_{i=1}^{n} C_{in} \exp(-\lambda_i t)$

where $C_{in} = \prod_{j \neq i} \frac{\lambda_i}{\lambda_j - \lambda_i}$

Hence $P(X(t) < j \mid X(0)=i) = \sum_{k=1}^{j-1} e^{-\lambda_k t} \prod_{r \neq k}^{j-1} \frac{\lambda_r}{\lambda_r - \lambda_k}$

and $P(X(t) < j+1 \mid X(0)=i) = \sum_{k=1}^{j} e^{-\lambda_k t} \prod_{\substack{r=i \\ r \neq k}}^{j} \frac{\lambda_r}{\lambda_r - \lambda_k}$

Hence we have the following

$P(X(t)=j \mid X(0)=i) = \sum_{k=1}^{j} e^{-\lambda_k t} \prod_{\substack{r=i \\ r \neq k}}^{j} \frac{\lambda_r}{\lambda_r - \lambda_k} - \sum_{k=1}^{j-1} e^{-\lambda_k t} \prod_{r \neq k}^{j-1} \frac{\lambda_r}{\lambda_r - \lambda_k}$

Proposition 6.1

Example, Yule process, i.e $\lambda_n = n\lambda$ $n \geq 1$

Let $i = 1$.

$P_{1j}(t) = e^{-j\lambda t} \prod_{r=1}^{j-1} \frac{r}{r-j} + \sum_{k=1}^{j-1} e^{-k\lambda t} \left[\prod_{r \neq k}^{j} \frac{r}{r-k} - \prod_{r \neq k}^{j-1} \frac{r}{r-k} \right]$

$= e^{-j\lambda t}(-1)^{j-1} + \sum_{k=1}^{j-1} e^{-k\lambda t} \left[\frac{j}{j-k} - 1 \right] \prod_{\substack{r=1 \\ r \neq k}}^{j-1} \frac{r}{r-k}$

$\frac{k}{j-k} \prod_{\substack{r=1 \\ r \neq k}}^{j-1} \frac{r}{r-k} = \frac{k}{j-k} \prod_{r=1}^{k-1} \frac{r}{r-k} \cdot \prod_{r=k+1}^{j-1} \frac{r}{r-k}$

$= \frac{(j-1)!}{(1-k)(2-k)\cdots(k-1-k)\,1\cdot 2\cdots(j-1-k)} \frac{1}{(j-k)} = \frac{(j-1)!}{(k-1)!(j-1)!} (-1)^{k-1}$

so $P_{1j}(t) = \sum_{k=1}^{j-1} \binom{j-1}{k-1} e^{-k\lambda t} (-1)^{k-1}$

$= e^{-\lambda t} \sum_{i=0}^{j-1} \binom{j-1}{i} e^{-i\lambda t} (-1)^i = e^{-\lambda t} (1 - e^{-\lambda t})^{j-1}$

which is the density of a geometric distribution.

Remember $(i=1)$, i.e population consists of 1 individual at time 0.

We are interested in the situation where $i > 1$. Using that a sum of geometric distributed variables is negative binomial and the fact that the population for i processes can be seen as the sum of population for each, one gets that the total sum will be negative binomial, i.e

$P_{ij}(t) = \binom{j-1}{i-1} e^{-i\lambda t} (1 - e^{-\lambda t})^{j-i}$ $j \geq i \geq 1$.

Now, look at a more general procedure to study $P_{ij}(t)$.

Define

$q_{ij} = v_i P_{ij}$, so q_{ij} is the rate for the process to make a transition from i to j

rate for transition out of i

q_{ij} is the instantaneous transition rates

Remark 1.41 • $v_i = \sum_j v_i P_{ij} = \sum_j q_{ij}$

•• $P_{ij} = \frac{q_{ij}}{v_i} = \frac{q_{ij}}{\sum_j q_{ij}}$

so the instantaneous transition rates determine the parameters of the process.