$P_{ij}(t) = P(X(s+t)=j \mid X(s)=i)$

transition probabilities.

$v_i \quad i=0,1,2\ldots$ instantaneous transition rates is an alternative expression

$P=\{P_{ij}\}$

$P_{ii}=0.$ $\qquad q_{ij} = v_i P_{ij}$

The transition probabilities satisfy some differential equations which are useful to determine $P_{ij}(t)$.

Lemma 6.2 a) $\lim_{h\to 0} \frac{1-P_{ii}(h)}{h} = v_i$

b) $\lim_{h\to 0} \frac{P_{ij}(h)}{h} = q_{ij}$

Proof. The probability for two or more transitions in $(0,h)$ is $o(h)$ since the time between transitions are independent and exponentially distributed. $1-P_{ii}(h)$ is the probability for not being in state i at time h. $1-P_{ii}(h)$ is therefore equal to the probability of one transition in $(0,h)$ + $o(h)$. Hence

$$1-P_{ii}(h) = v_i h + o(h)$$

$$\lim_{h\to 0} \frac{1-P_{ii}(h)}{h} = v_i \quad \text{which is a)}$$

For b): $P_{ij}(h)$ is the probability that an event has occurred and a transition from i to j, $j\neq i$ has taken place in $(0,h)$

so $P_{ij}(h) = \underbrace{v_i P_{ij}}_{q_{ij}} h + o(h)$

so $\lim_{h\to 0} \frac{P_{ij}(h)}{h} = q_{ij} \quad i\neq j$ ∎

The analogue to the Chapman-Kolmogorov equations for discrete times Markov chains is the following for continuous time.

Lemma 6.2 For all $s\geq 0$, $t\geq 0$

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t)\cdot P_{kj}(s)$$

Proof

$P_{ij}(t+s) = P(X(t+s)=j \mid X(0)=i)$

$= \sum_{k=0}^{\infty} P(X(t+s)=j, X(t)=k \mid X(0)=i)$

$P(A\cap B) = P(A|B)P(B)$

$= \sum_{k=0}^{\infty} P(X(t+s)=j \mid X(t)=k, X(0)=i)\cdot P(X(t)=k \mid X(0)=i)$

$= \sum_{k=0}^{\infty} P(X(t+s)=j \mid X(t)=k)\, P(X(t)=k \mid X(0)=i)$

$= \sum_{k=0}^{\infty} P_{ik}(t)\, P_{kj}(s).$ ∎

These results can be used to derive the partial differential equations.

Comparing time $t+h$ and t

$$P_{ij}(t+h) - P_{ij}(t) = \sum_{k=0}^{\infty} P_{ik}(h)\cdot P_{kj}(t) - P_{ij}(t)$$

$$= \sum_{\substack{k=0\\k\neq i}}^{\infty} P_{ik}(h)\cdot P_{kj}(t) - [1-P_{ii}(h)]P_{ij}(t)$$

or

$$\frac{P_{ij}(t+h)-P_{ij}(t)}{h} = \sum_{k\neq i} \frac{P_{ik}(h)}{h}\cdot P_{kj}(t) - \frac{[1-P_{ii}(h)]}{h} P_{ij}(t)$$

lemma 6.2

If limit and summation can be interchanged the result is the Kolmogorov backward equation

Theorem 6.1 $P_{ij}'(t) = \sum_{k\neq i} q_{ik}\cdot P_{kj}(t) - v_i P_{ij}(t)$ for all $i,j\geq 0$, $t\geq 0$

Example, pure birth process.

Backward eqn.

$v_i = \lambda_i \qquad q_{ij} = v_i P_{ij} = \lambda_i \quad i=0,1,2\ldots \quad j=i+1$

$P_{ij} = \begin{cases} 1 & j=i+1 \\ 0 & \text{else} \end{cases}$

so $P_{ij}'(t) = \sum_{k\neq i} q_{ik}\cdot P_{kj}(t) - v_i P_{ij}(t)$

$= q_{i,i+1}\cdot P_{i+1,j}(t) - \lambda_i P_{ij}(t).$

Example birth and death process.

$v_0 = \lambda_0$

$v_i = \lambda_i + \mu_i \quad i>0 \qquad q_{01} = v_0 P_{01} = \lambda_0$

$P_{01} = 1$

$P_{i,i+1} = \frac{\lambda_i}{\lambda_i+\mu_i} \quad i>1 \qquad q_{ij} = v_i P_{ij} = \begin{cases} (\lambda_i+\mu_i)\cdot\frac{\lambda_i}{\lambda_i+\mu_i} = \lambda_i \\ (\lambda_i+\mu_i)\cdot\frac{\mu_i}{\lambda_i+\mu_i} = \mu_i \end{cases}$

$P_{i,i-1} = \frac{\mu_i}{\lambda_i+\mu_i}$

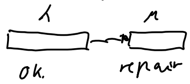
so the backward equation is

$P_{0j}'(t) = q_{01}\cdot P_{1j}(t) - v_0 P_{0j}(t)$

$= \lambda_0 \cdot P_{1j}(t) - \lambda_0 P_{0j}(t)$

$P_{ij}'(t) = \lambda_i \cdot P_{i,i+1}(t) + \mu_i P_{i,i-1}(t) - (\lambda_i+\mu_i)P_{ij}(t)$

Example: two stage continuous time Markov chain



$X(t) = \begin{cases} 0 & \text{machine working} \\ 1 & \text{repaired} \end{cases}$

$X(0) = 0$

$P_{00}(0) = 1, \quad P_{01}(t=10) = ?$

This is a birth and death process

$\lambda_0 = \lambda \qquad \mu_1 = \mu \qquad | \quad q_{01} = \lambda$

$\lambda_i = 0 \;\; i \neq 0 \qquad \mu_i = 0 \;\; i \neq 1 \qquad | \quad q_{10} = \mu$

Backwards equations:

$(*) \quad P_{00}'(t) = \lambda \cdot P_{10}(t) - \lambda \cdot P_{00}(t) \quad | \; \mu$

$P_{10}'(t) = \mu P_{00}(t) - \mu P_{10}(t) \quad | \; \lambda$

Multiplying and adding

$\mu \cdot P_{00}'(t) + \lambda \cdot P_{10}'(t) = 0$

so $\mu P_{00}(t) + \lambda \cdot P_{10}(t) = c$

Use $P_{00}(0) = 1$, $P_{10}(0) = 0$ so

$c = \mu \Rightarrow$

$\lambda \cdot P_{10}(t) = \mu [1 - P_{00}(t)]$

Substitute in $(*)$

$P_{00}'(t) = \mu [1 - P_{00}(t)] - \lambda P_{00}(t)$

$= \mu - (\mu+\lambda) P_{00}(t) = -(\mu+\lambda) \cdot h(t)$

$h(t) = P_{00}(t) - \frac{\mu}{\mu+\lambda}$

$h'(t) = \mu - (\mu+\lambda) \cdot [h(t) + \frac{\mu}{\mu+\lambda}]$

$= -(\mu+\lambda) \cdot h(t)$

$\Rightarrow \frac{\partial}{\partial t} \log(h(t)) = -(\mu+\lambda)$

$\log h(t) = -(\mu+\lambda)t + c$

$h(t) = K \cdot e^{-(\mu+\lambda)t}$

so $P_{00}(t) = \frac{\mu}{\mu+\lambda} + K \cdot e^{-(\mu+\lambda)t}$

$P_{00}(0) = 1 \Rightarrow K = \frac{\lambda}{\mu+\lambda} \Rightarrow P_{00}(t) = \frac{\lambda}{\mu+\lambda} e^{-(\mu+\lambda)t} + \frac{\mu}{\mu+\lambda}$

$P_{10}(t) = \frac{\mu}{\lambda} [1 - P_{00}(t)]$

$= \frac{\mu}{\lambda} \left[\frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\mu+\lambda)t} \right]$

$= \frac{\mu}{\mu+\lambda} - \frac{\mu}{\mu+\lambda} e^{-(\mu+\lambda)t}$

$P_{00}(10)$ inserting in $P_{00}(t)$.

Forward equations ? By Chapman-Kolmogorov eqns

$P_{ij}(t+h) - P_{ij}(t) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(h) - P_{ij}(t)$

$= \sum_{k \neq j} P_{ik}(t) P_{kj}(h) - [1 - P_{jj}(h)] P_{ij}(t)$

so $\frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \sum_{k \neq j} \underset{\downarrow\, q_{kj}}{\frac{P_{kj}(h)}{h}} P_{ik}(t) - \underset{\downarrow\, v_j}{\frac{1 - P_{jj}(h)}{h}} P_{ij}(t)$

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so if interchange of limit and summation is permitted we get the Kolmogorov forward equation

Theorem 6.2

$P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t)$

Example pure birth process

$v_i = \lambda_i \quad 0, 1, 2 \ldots$

$v_i - \lambda_i \quad i \geq 0 \qquad q_{i,i+1} = \lambda_i \quad i = 0, 1, 2 \ldots$

$P_{i,i+1} = 1$

$P_{ij}'(t) = q_{j-1,j} P_{i,j-1}(t) - v_j P_{ij}(t)$

$= \lambda_{j-1} P_{i,j-1}(t) - \lambda_j P_{ij}(t)$

$= \begin{cases} -\lambda_i P_{ii}(t) & i = j \quad \text{since } P_{ij}(t) = 0 \;\; i > j \\ \lambda_{j-1} P_{i,j}(t) - \lambda_j P_{ij}(t) & j = i+1, \ldots \end{cases}$

Multiplying with $e^{\lambda_i t}$

$P_{ii}(t) = e^{-\lambda_i t} \quad i \geq 0$

and $e^{\lambda_j t} [P_{ij}'(t) + \lambda_j P_{ij}(t)] = e^{\lambda_j t} [\lambda_{j-1} P_{i,j-1}(t)]$

or $\frac{\partial}{\partial t} [e^{\lambda_j t} P_{ij}(t)] = e^{\lambda_j t} \lambda_{j-1} P_{i,j-1}(t)$

So

Proposition 6.4. For a pure birth process

$P_{ii}(t) = e^{-\lambda_i t} \quad i \geq 0$

$P_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{\lambda_j s} P_{i,j-1}(s)\, ds \quad j \geq i+1$