Transition probabilities $P_{ij}(t)$

Instantanous transition probabilities q_{ij}

Kolmogorov forward eqn.

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t)$$

Example: Birth and death process

$v_0 = \lambda_0$ $\qquad q_{i-1,i} = \lambda_{i-1}$

$v_i = \lambda_i + \mu_i$ $\qquad q_{i,i-1} = \mu_i$

$$P_{ij} = \begin{cases} \frac{\lambda_i}{\lambda_i + \mu_i} \\ \frac{\mu_i}{\lambda_i + \mu_i} \end{cases}$$

$P_{01} = 1$

Forward eqn $\quad P'_{i0}(t) = \sum_{k \neq 0} q_{k0} P_{ik}(t) - \lambda_0 P_{i0}(t)$

$$= \mu_1 P_{i1}(t) - \lambda_0 P_{i0}(t)$$

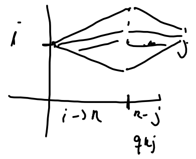
$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - (\lambda_j + \mu_j) P_{ij}(t)$$

$$= \lambda_{j-1} P_{i,j-1}(t) + \mu_i P_{i,j+1}(t) - (\lambda_j + \mu_j) P_{ij}$$

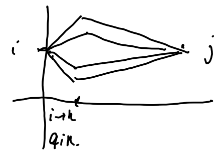
Forward eqn: $P'_{ij}(t) = \underbrace{\sum_{k \neq j} q_{kj} P_{ik}(t)}_{\sum_{k \neq j} P_{ik}(t) q_{kj}} - v_j P_{ij}(t)$

Backward eqn $\quad P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$

Forward



Backward



6.5 Limit probabilities

Often $\lim_{t \to \infty} P_{ij}(t) = P_j$ exists and is independent of the initial state.

Consider the forward eqn.

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik} - v_j P_{ij}(t)$$

Let $t \to \infty$, and assume that summation and limit can be interchanged

$$\lim_{t \to \infty} P'_{ij}(t) = \lim_{t \to \infty} \left[\sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t) \right]$$

$$= \sum_{k \neq j} q_{kj} P_k - v_j P_j$$

$0 \leq P_{ij}(t) \leq 1$ so bounded, so if $\lim_{t \to \infty} P'_{ij}(t)$ exists it must be 0 else the $P_{ij}(t)$ cannot be bounded. Hence $\boxed{v_j P_j = \sum_{k \neq j} q_{kj} P_k}$ $\forall j$

Also $\boxed{\sum_j P_j = 1}$ and the limiting probabilities $\{P_j\}$ can be found.

Remark 6.5.1 Sufficient conditions for $\lim_{t \to \infty} P_{ij}(t)$ to exist is

a) The states communicate, $\forall i,j$ $P_{ij}(t) > 0$ some t.

b) The Markov chain is recurrent, so starting in a certain state, the expected time to return to the state is finite.

Under these conditions (a + b), the "balance equations"

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k \quad \text{and} \quad \sum_j P_j = 1$$

are satisfied.

Remark 6.5.2 $v_j P_j$ can be interpreted as the rate at which the process leaves state j

$\sum_{k \neq j} q_{kj} P_k$ can be interpreted as the rate at which the process enters state j

The "balance equations" state that the two rates are equal.

Remark 6.5.3 The Markov chain is ergodic when $\lim_{t \to \infty} P_{ij}(t) = P_j$ exists.

The P_j's are called the <u>stationary probabilities</u>. If the chain is started according $\{P_j\}$ then one can show

$P(X(t) = j) = P_j \quad \forall t$.

P_j for birth and death processes?

Can use the balance equations

State	rate for leaving		rate entering
0	$\lambda_0 P_0$	=	$\mu_1 P_1$
1	$(\lambda_1 + \mu_1) P_1$	=	$\mu_2 P_2 + \lambda_0 P_0$
2	$(\lambda_2 + \mu_2) P_2$	=	$\mu_3 P_3 + \lambda_1 P_1$

so, adding ~~the~~ equation to the preceding one we get.

$$\lambda_0 P_0 = \mu_1 P_1$$
$$\lambda_1 P_1 = \mu_2 P_2$$
$$\lambda_2 P_2 = \mu_3 P_3$$
$$\vdots \qquad \vdots$$
$$\lambda_n P_n = \mu_{n+1} P_{n+1}$$

so

$$P_1 = \frac{\lambda_0}{\mu_1} P_0$$

$$P_2 = \frac{\lambda_1}{\mu_2} P_1 = \frac{\lambda_1 \lambda_0}{\mu_1 \mu_2} \cdot P_0$$

$$\vdots$$

$$P_n = \frac{\lambda_{n-1}}{\mu_n} P_{n-1} = \cdots \frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1} \cdot P_0$$

using $\sum P_j = 1$,

$$P_n = \frac{\lambda_0 \cdots \lambda_n}{\mu_1 \cdots \mu_n} \underbrace{\left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} \right)}_{P_0} \quad n \geq 1$$

A <u>necessary</u> condition for $\{P_j\}$ to exist is

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} < \infty.$$

Example $M/M/s$ s servers

$$\sum_{n=s+1}^{\infty} \frac{\lambda^n}{(s\mu)^n} < \infty$$

when $\frac{\lambda}{s\mu} < 1$

Example, linear birth and death with immigration

$\lambda_n = \theta + (n-1)\lambda, \quad \mu_n = n\mu$

$$\sum_{n=1}^{\infty} \frac{\theta(\theta+\lambda)\cdots(\theta+(n-1)\lambda)}{\mu \cdot 2\mu \cdots n\mu} < \infty \ ?$$

Ratio test.

$$\lim_{n\to\infty} \frac{\theta(\theta+\lambda)\cdots(\theta+n\lambda)}{(n+1)!\,\mu^{n+1}} \cdot \frac{n!\,\mu^n}{\theta(\theta+\lambda)\cdots\theta+(n-1)\lambda}$$

$$= \lim_{n\to\infty} \frac{\theta+n\lambda}{(n+1)\cdot\mu} = \frac{\lambda}{\mu}.$$

so $\{P_j\}$ exists when $\lambda < \mu$.

Example, machine repair.

M machines, one repairman

Failure rate each machine λ.

Time to repair $\mu\exp(-\mu x)$, i.e. mean/expectation $1/\mu$.

Average number of machines working ?

Proportion of time a certain machine — " — ?

State $X(t) = n$ when n machines not in use i.e. $M-n$ in use.

This is a birth & death process

$\mu_n = \mu$

$$\lambda_n = \begin{cases} (M-n)\lambda & n \le M \\ 0 & n > M \end{cases}$$

$$P_0 = \frac{1}{1 + \sum_{n=1}^{M} \left(\frac{\lambda}{\mu}\right)^n \cdot M(M-1)\cdots(M-n+1)}$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{M!}{(M-n)!} \left(\frac{1}{1 + \sum_{k=1}^{M} \left(\frac{\lambda}{\mu}\right)^k \cdot \frac{M!}{(M-k)!}} \right)$$

so the average number of machines not in use is

$$\underline{\underline{\sum_{n=0}^{M} n \cdot P_n}} = \sum_{n=0}^{M} n \cdot \frac{\left(\frac{\lambda}{\mu}\right)^n \cdot \frac{M!}{(M-n)!}}{1 + \sum_{n=1}^{M} \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{M!}{(M-n)!}}$$

— P_n

Long run proportion of time a given machine is ~~not~~ working equals.

$P(\text{machine is working}) =$

$$\sum_{n=0}^{M} P(\text{machine is working} \mid n \text{ not working}) \cdot P_n$$

$$= \sum_{n=0}^{M} \frac{M-n}{M} P_n = 1 - \sum_{n=0}^{M} \frac{n \cdot P_n}{M} = \underline{\underline{1 - \frac{1}{M} \cdot \sum_{n=0}^{M} n \cdot P_n}}$$

Example $M/M/1$ queue.

State $X(t) = n$ number in system

$\lambda_n = \lambda$

$\mu_n = \mu$

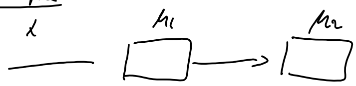
$$P_n = \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n} = \left(\frac{\lambda}{\mu}\right)^n \cdot \left(1 - \frac{\lambda}{\mu}\right) \quad n \ge 0$$

$\{P_n\}$ exists when $\frac{\lambda}{\mu} < 1$, i.e. arrival rates less than service rates

$\left(\frac{\lambda}{\mu}\right) = 1$

What about continuous time Markov chains that are not birth & death processes

__Example__



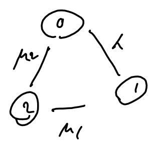
both servers must be free for service to start.

state

0	no present
1	1 occupied, 2 free
2	1 free, 2 occupied

$v_0 = \lambda_1, \ v_1 = \mu_1, v_2 = \mu_2$

Balance equ's



State	enter		Leave
0	$P_2\mu_2$	$=$	$P_0\lambda$
1	$P_0\lambda$	$=$	$P_1\mu_1$
2	$P_1\mu_1$	$=$	$P_2\mu_2$

$\Rightarrow P_2 = \frac{\lambda}{\mu_2} P_0$

$P_1 = \frac{\lambda}{\mu_1} P_0$

So $$P_0 = \frac{1}{1 + \frac{\lambda}{\mu_2} + \frac{\lambda}{\mu_1}} = \frac{\mu_1\mu_2}{\mu_1\mu_2 + \lambda(\mu_1+\mu_2)}$$

$$P_1 = \frac{\lambda\mu_2}{\mu_1\mu_2 + \lambda(\mu_1+\mu_2)}$$

$$P_2 = \frac{\lambda \cdot \mu_1}{\mu_1\mu_2 + \lambda(\mu_1+\mu_2)}.$$