Example: n components, one repairman

Functioning time component $i \sim \lambda_i \exp(-\lambda_i x)$

Repair time component $i \sim \mu_i \exp(-\mu_i x)$

Repairman always works on the most recent failure.

Must take into account the order of failure of failed comp[o]nents.

State $X(t) = i_1 \cdots i_k$ which is the failed components in failure order.

Number of states: $\sum_{k=0}^{n} \binom{n}{k} \cdot k! = n! \sum_{i=0}^{n} \frac{1}{i!}$

$N(n)$

n	$N(n)$
2	5
3	40
4	84

Balance equations:

Leaving state $i_1, \dots, i_k$: $\left(\mu_{i_1} + \sum_{\substack{i \neq i_j \\ j=1,\dots,k}} \lambda_i\right) P(i_1, \dots, i_k)$

repair of i_1 — failure of a functioning component

Entering state $i_1, \dots, i_k$

$\sum_{\substack{i \neq i_j \\ j=1,\dots,k}} [\mu_i P(i, i_1, \dots, i_k)] + \lambda_{i_1} P(i_2, \dots, i_k)$

failure of component i_k when state is $i_2, \dots, i_k$

One additional equation necessary

$$\sum_{i=1}^{n} \lambda_i P(\phi) = \sum_{i=1}^{n} P(i) \cdot \mu_i$$

leaving state ϕ — entering state ϕ

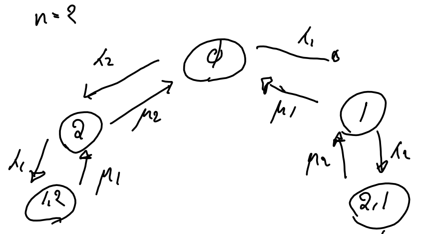
all components O.K.

One can see that the balance eq'n's has solution

$$P(i_1, \dots, i_k) = \frac{\lambda_{i_1} \cdots \lambda_{i_k}}{\mu_{i_1} \cdots \mu_{i_k}} \cdot P(\phi)$$

$$\text{so } P(\phi) = \frac{1}{1 + \sum_{i_1, \dots, i_k} \frac{\lambda_{i_1} \cdots \lambda_{i_k}}{\mu_{i_1} \cdots \mu_{i_k}}}$$

$n = 2$



state	leaving	entering
ϕ	$P(\phi)(\lambda_1 + \lambda_2)$	$\mu_1 P(1) + \mu_2 P(2)$
1	$P(1)(\lambda_2 + \mu_1)$	$\lambda_1 P(\phi) + \mu_2 P(2,1)$
2	$P(2)(\lambda_1 + \mu_2)$	$\lambda_2 P(\phi) + \mu_1 P(1,2)$
2 1	$\mu_2 P(2,1)$	$\lambda_2 P(1)$
1 2	$\mu_1 P(1,2)$	$\lambda_1 P(2)$

$= P(2,1)$

Solving $P(1)(\lambda_2 + \mu_1) = \lambda_1 P(\phi) + \mu_2 \left[\frac{\lambda_2}{\mu_2} P(1)\right]$

$P(1) = \frac{\lambda_1}{\mu_1} P(\phi) \Rightarrow P(2,1) = \frac{\lambda_2}{\mu_2} \frac{\lambda_1}{\mu_1} P(\phi)$

6.8 Uniformization

Consider the situation where $v_i = v$ all states i. Then $\{N(t);\ t \geq 0\}$ will be a Poisson process with rate v.

What happens with the transition probabilities if we condition on $N(t)$.

$$P_{ij}(t) = P(X(t) = j \mid X(0) = i)$$
$$= \sum_{n=0}^{\infty} P(X(t) = j \mid X(0) = i, N(t) = n) \cdot \underbrace{P(N(t) = n \mid X(0) = i)}_{\frac{(vt)^n}{n!} e^{-vt}}$$

Knowledge about the number of transitions contains information about the time spent. Since the distribution of time spent in each state is the same, there is no information about which state was visited.

Hence when $v_i = v$

$$P_{ij}(t) = \sum_{n=0}^{\infty} P_{ij}^{n} \cdot e^{-vt} \frac{(vt)^n}{n!}$$

which can used for computing $P_{ij}(t)$

But the result can also be used when $v_i \leq v$ all i by allowing fictitious transitions from a state to itself with probability $1 - \frac{v_i}{v}$ when the change / transition is generated with rate v.