Uniformization.

Last time, if all $v_i = v$, then conditioning on number of events

$$P_{ij}(t) = \sum_{n=0}^{\infty} P_{ij}^{n}\, e^{-vt} \cdot \frac{(vt)^n}{n!}$$

which can be used to compute numeric values for $P_{ij}(t)$.

The result can also be useful when $v_i \le v \quad \forall i$ by allowing fictitious transitions from a state to itself with probability $1-\frac{v_i}{v}$.

Leaving a state with rate v_i can be thought as having transitions at rate v but only $\frac{v_i}{v}$ are real and a fraction $1-\frac{v_i}{v}$ just remain in state i.

Thus a continuous time Markov chain with rate v_i where $v_i \le v \ \forall i$ has the same distribution as a Markov chain with rate v where the jump matrix is

$$P_{ij}^{*} = \begin{cases} 1-\frac{v_i}{v} & \text{(fictitious transitions)} \\ \frac{v_i}{v} & \text{(real transitions)} \end{cases}$$

Thus $P_{ij}(t) = \sum_{n=0}^{\infty} P_{ij}^{*n}\, e^{-vt} \cdot \frac{(vt)^n}{n!}$

Example, machine repair

state
- 0 on, breakdown rate $v_0 = \lambda$
- 1 off, repair rate $v_1 = \mu$

$P_{01} = P_{10} = 1$.

Take $v = \lambda + \mu$

Then $P_{0j}^{*} = \begin{cases} 1-\frac{v_0}{v}, & j=0 \\ \frac{v_0}{v} \cdot P_{01}, & j=1 \end{cases} = \begin{cases} \frac{\mu}{\mu+\lambda} & j=0 \\ \frac{\lambda}{\lambda+\mu} & j=1 \end{cases}$

$P_{1j}^{*} = \begin{cases} 1-\frac{v_1}{v}, & j=1 \\ \frac{v_1}{v} P_{10}, & j=0 \end{cases} = \begin{cases} \frac{\lambda}{\lambda+\mu} & j=1 \\ \frac{\mu}{\mu+\lambda} & j=0 \end{cases}$

Hence $P_{00}^{*} = P_{10}^{*} = \frac{\mu}{\mu+\lambda}$

$P_{11}^{*} = P_{01}^{*} = \frac{\lambda}{\lambda+\mu}$

The transition into 0 (or 1) does not depend on the present state, so

$P_{i0}^{*n} = \frac{\mu}{\mu+\lambda}$ and $P_{i1}^{*n} = \frac{\lambda}{\lambda+\mu}$ when $n \ge 1$, $i = 0,1$

Hence

$$P_{00}(t) = \sum_{n=0}^{\infty} P_{00}^{*n}\, e^{-(\lambda+\mu)t} \frac{[t(\lambda+\mu)]^n}{n!}$$

$$= e^{-(\lambda+\mu)t} + \frac{\mu}{\mu+\lambda} \cdot e^{-(\lambda+\mu)t} \underbrace{\sum_{n=1}^{\infty} \frac{[(\lambda+\mu)t]^n}{n!}}_{1-\exp((\lambda+\mu)t)}$$

$$= e^{-(\lambda+\mu)t} \cdot \frac{\mu}{\mu+\lambda} \cdot [1 - e^{-(\lambda+\mu)t}]$$

$$= \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda} e^{-(\lambda+\mu)t}$$

as before, cfr Ross p. 372.

Similarly

$$P_{11}(t) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}[1 - \exp(-t(\mu+\lambda))]$$

$P_{01}(t) = 1 - P_{00}(t)$

$P_{10}(t) = 1 - P_{11}(t)$

Let $O(t)$ = time spent in state 0 in interval $(0,t)$

$$E[O(t)] = \int_{0}^{t} E\{I[X(s)=0]\}\, ds$$

$$= \int_0^t P(X(s)=0)\, ds$$

$$= \int_0^t \left[\frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda} \cdot e^{-(\lambda+\mu)s}\right] ds$$

$$= \frac{\mu}{\mu+\lambda} t + \frac{\lambda}{(\lambda+\mu)^2}[1 - e^{-(\lambda+\mu)t}].$$

6.9. Computing the transition probabilities

Introduce $r_{ij} = \begin{cases} q_{ij} & i \ne j \\ -v_i & i=j \end{cases}$

Kolmogorov backward equation

$$P_{ij}'(t) = \sum_{k \ne i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

$$= \sum_k r_{ik} P_{kj}(t)$$

forward equations

$$P_{ij}'(t) = \sum_{k \ne j} q_{kj} P_{ik}(t) - v_j P_{ij}(t) = \sum r_{kj} P_{ik}(t)$$

Define matrices R, $P(t)$, $P'(t)$ having (i,j) elements equal to r_{ij}, $P_{ij}(t)$, $P_{ij}'(t)$

Backward equation: $P'(t) = R\, P(t)$

Forward — " — $P'(t) = P(t) \cdot R$

R: infinitesimal generator matrix

$f'(t) = f(t) \quad f = c e^{t}$

Solution of the equations are

$P(t) = P(0) \cdot e^{Rt} = e^{Rt}$ since $P(0) = I$

where $e^{Rt} = \sum_{n=0}^{\infty} R^n \frac{t^n}{n!}$

Direct use of $P(t) = e^{Rt}$ can be complicated numerically since R contain negative values.

But there are alternatives.