Chapter 10 Brownian motion.

Here we consider a Markov process in continuous time, when in addition the state is $\mathbb{R}$, i.e. not even countable.

It can be seen as a limit of a symmetric random walk when the step size and the time units decrease.

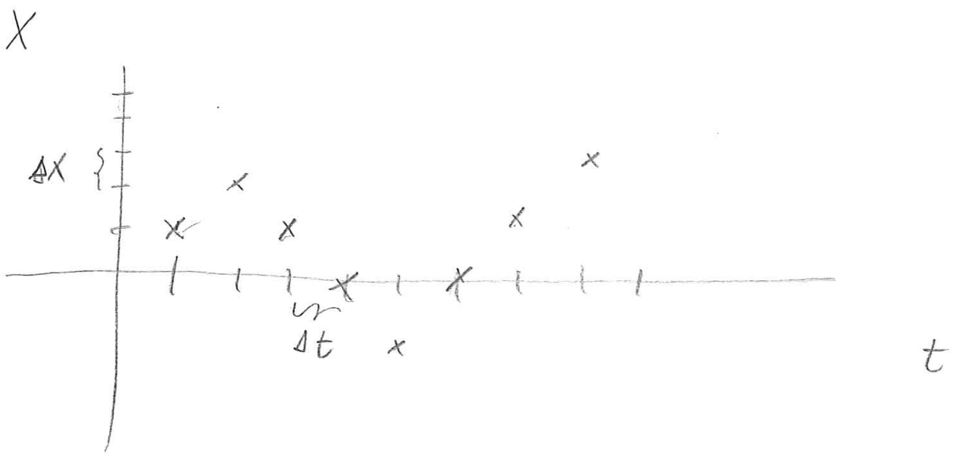
More specific. Let Δt be time unit and ΔX step size
so the position at time t, $X(t)$ is

$$X(t) = \Delta X(X_1 + \cdots + X_{[t/\Delta t]}) \quad ([\,\cdot\,] \text{ integer value})$$

when $X_i = \begin{cases} +1 & \text{step up size } \Delta X \\ -1 & \text{step down} \text{ — " —} \end{cases}$

Assume $X_1, X_2, \ldots$ independent $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$

$$X\left(\frac{s}{\Delta t}\right) - X\left(\frac{s-1}{\Delta t}\right) = \Delta X \cdot X_{[s/\Delta t]} \qquad 0 \le s \le t$$



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Now $E(X_i) = 0$, $Var(X_i) = E(X_i^2) = 1$

so $E[X(t)] = 0$

$Var(X(t)) = (\Delta x)^2 \cdot \left[\frac{t}{\Delta t}\right]$

If $\Delta t, \Delta X \to 0$, so that $\Delta X = \sigma\sqrt{\Delta t}$

$E[X(t)] = 0$

$Var(X(t)) \to \sigma^2 \cdot t$

The following can be shown to be true.

(i) By the central limit theorem
$X(t) \sim N(0, \sigma^2 t)$ approximately
since $X(t)$ sum of independent variables

(ii) $\{X(t)\ t \geq 0\}$ has independent increments
since if $t_1 < t_2 < \dots < t_n$
$X(t_n) - X(t_{n-1})$, $X(t_{n-1}) - X(t_{n-2})$, …, $X(t_2) - X(t_1)$, $X(t_1)$
are independent depending on disjunct sets of variables.

(iii) The distribution of $X(t+s) - X(t)$ depends only on s, i.e the number of units in $(s, s+t)$,
so the $X(t)$ has stationary increments.

This motivates the following formal definition

<u>Definition 10.1</u> A stochastic process $\{X(t), t \geq 0\}$ is a <u>Brownian motion</u> process / Wiener process.

(i) $X(0) = 0$

(ii) $\{X(t), t \geq 0\}$ has stationary independent increments

(iii) If $t > 0$, $X(t) \sim N(0, \sigma^2 t)$.

Remark 10.1.1
$\sigma = 1$: Standard Brownian motion, since $X(t) \sim N(0, t \cdot \sigma^2)$
$\Rightarrow B(t) = \frac{X(t)}{\sigma^2} \sim N(0, t)$,
assume from now on $\sigma^2 = 1$

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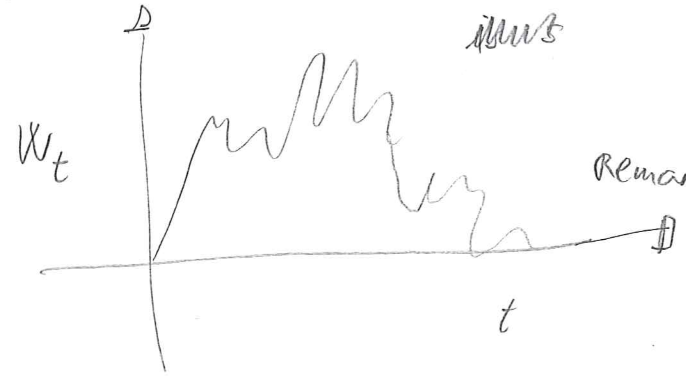

Remark 10.1.2
Application: movement of particles in liquids or gas. Einstein explained movement as result of bombardment of molecules.

Two properties.

- $E[X(t+h) - X(t)] = 0$

  $E[\qquad\qquad]^2 = h.$

If $h \to 0$ this indicates that

$X(t+h) - X(t)$ converges to zero and $X(t)$ is continuous. This result can be shown rigorously.

- $E\left[\frac{X(t+h) - X(t)}{h}\right] = 0$

  $E\left[\frac{X(t+h) - X(t)}{h}\right]^2 = \frac{h}{h^2} = \frac{1}{h}$

Thus we cannot conclude that $\frac{X(t+h) - X(t)}{h}$ converges to 0 as $h \to 0$, indications that $X(t)$, $t \geq 0$ is not differentiable.

Remember one dimensional transformation

$U = g(V)$ g strictly increasing

$$F_U(U \le u) = P(U \le u) = P(g(V) \le u)$$
$$= P(V \le g^{-1}(u)) = F_V(g^{-1}(u))$$

so $f_U(u) = f_V(g^{-1}(u)) \cdot \frac{\partial}{\partial u} g^{-1}(u)$

In general if $\underset{\sim}{U}, \underset{\sim}{V} \in \mathbb{R}^n$

$f_{\underset{\sim}{U}}(\underset{\sim}{u}) = f_V(g^{-1}(u)) \cdot |J|$, $|J|$ Jacobian of g^{-1}

Let $\underset{\sim}{U} = \begin{pmatrix} X(t_1) \\ \\ X(t_n) \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots \\ 1 & \cdots & & 1 \end{pmatrix} \begin{pmatrix} X(t_1) \\ X(t_2) - X(t_1) \\ \\ X(t_n) - X(t_{n-1}) \end{pmatrix} = G \cdot \underset{\sim}{V}$

$$G^{-1}\underset{\sim}{U} = G^{-1}\begin{pmatrix} X(t_1) \\ \\ X(t_n) \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & 0 \cdots & 0 \\ 0 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} X(t_1) \\ \vdots \\ X(t_n) \end{pmatrix} = \begin{pmatrix} X(t_1) \\ X(t_2) - X(t_1) \\ \\ X(t_n) - X(t_{n-1}) \end{pmatrix} = \underset{\sim}{V}$$

f_U $f_V(g^{-1}u)$ J

so $f(x_1, \dots, x_n) = f_{t_1}(x_1) \cdot f_{t_2 - t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n - x_{n-1}) \cdot 1$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\tfrac{1}{2}x_1^2\right) \exp\left(-\tfrac{1}{2}\frac{(x_2 - x_1)^2}{t_2 - t_1}\right) \cdots \exp\left(-\tfrac{1}{2}\frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}\right)$$

so $(X_1, \dots, X_n)'$ is multinormally distributed